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DOUBLE RESONANCE FOR ROBIN PROBLEMS WITH INDEFINITE AND UNBOUNDED POTENTIAL

Nikolaos S. Papageorgiou

National Technical University, Department of Mathematics
Zografou Campus
Athens 15780, Greece

Patrick Winkert

Technische Universität Berlin, Institut für Mathematik
Straße des 17. Juni 136
10623 Berlin, Germany

Abstract. We study a semilinear Robin problem driven by the Laplacian plus an indefinite and unbounded potential term. The nonlinearity $f(x,s)$ is a Carathéodory function which is asymptotically linear as $s \to \pm\infty$ and resonant. In fact we assume double resonance with respect to any nonprincipal, nonnegative spectral interval $\left[\hat{\lambda}_k, \hat{\lambda}_{k+1}\right]$. Applying variational tools along with suitable truncation and perturbation techniques as well as Morse theory, we show that the problem has at least three nontrivial smooth solutions, two of constant sign.

1. Introduction.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following semilinear Robin problem

$$\begin{aligned} -\Delta u + \xi(x)u &= f(x,u) \qquad \text{in } \Omega,\\ \frac{\partial u}{\partial n} + \beta(x)u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\xi : \Omega \to \mathbb{R}$ is a potential function being in general indefinite (that is, sign changing) and unbounded (more precisely, $\xi \in L^q(\Omega)$ for $q > N$). The nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $x \mapsto f(x,s)$ is measurable for all $s \in \mathbb{R}$ and $s \mapsto f(x,s)$ is continuous for a.a. $x \in \Omega$. In addition, in the boundary condition of (1.1), $\frac{\partial u}{\partial n}$ denotes the normal derivative defined by extension of the linear map

$$u \to \frac{\partial u}{\partial n} = (\nabla u, n)_{\mathbb{R}^N} \quad \text{for all } u \in C^1(\overline{\Omega}),$$

with $n : \Omega \to \mathbb{R}$ being the outward unit normal on $\partial\Omega$. The boundary coefficient β belongs to $W^{1,\infty}(\partial\Omega)$ satisfying $\beta(x) \geq 0$ for all $x \in \partial\Omega$. When $\beta = 0$, we recover the Neumann problem.

In this paper we assume that $f(x,\cdot)$ is asymptotically linear as $s \to \pm\infty$ and resonant, that is, the quotient $\frac{f(x,s)}{s}$ interacts with the spectrum of the operator

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$u \to -\Delta u + \xi(x)u$ with Robin boundary condition as $s \to \pm\infty$. However, the resonance assumed here is more general since we do not assume that the limits

$$\lim_{s\to\pm\infty} \frac{f(x,s)}{s}$$

exist, as is the case in most papers dealing with resonant equations, see, for example, Li-Liu [10] and Gasiński-Papageorgiou [8]. Instead, we assume here that we can have a so-called double resonance situation, namely

$$\hat{\lambda}_k \le \liminf_{s\to\pm\infty} \frac{f(x,s)}{s} \le \limsup_{s\to\pm\infty} \frac{f(x,s)}{s} \le \hat{\lambda}_{k+1},$$

with $\hat{\lambda}_k, \hat{\lambda}_{k+1}$ being two successive eigenvalues of the Robin differential operator. Such resonant problems were investigated almost exclusively in the context of Dirichlet boundary value problems with no potential term, that is, $\xi \equiv 0$. In this setting, the differential operator (left-hand side of the operator) is coercive and this simplifies significantly the analysis of the problem. The first work on doubly resonant equations is the one of Berestycki-de Figueiredo [3] who also coined the term double resonance. Subsequently appeared the related works of Các [4], Liang-Su [11], Robinson [16], Su [17] and Zou [19] who proved multiplicity results but under more restrictive conditions on the data of the Dirichlet problem. More precisely, Các [4] and Zou [19] do not allow complete resonance to occur at the two endpoints of the spectral interval $\left[\hat{\lambda}_k, \hat{\lambda}_{k+1}\right]$ while Liang-Su [11] and Robinson [16] assume that $f \in C\left(\overline{\Omega} \times \mathbb{R}\right)$. We also mention the recent works of Papageorgiou-Rădulescu [13], [15] who deal with resonant Neumann problems but do not address the case of double resonance.

Our approach is variational based on critical point theory together with suitable truncation and perturbation techniques and the usage of Morse theory in terms of critical groups. In the next section we develop the necessary mathematical background material which will help to follow the arguments in this paper.

2. **Preliminaries.** Let X be a Banach space and X^* its topological dual while $\langle \cdot, \cdot \rangle$ denotes the duality brackets to the pair (X^*, X). We have the following definition.

Definition 2.1. The functional $\varphi \in C^1(X, \mathbb{R})$ fulfills the Cerami condition (the C-condition for short) if the following holds: every sequence $(u_n)_{n\ge1} \subseteq X$ such that $(\varphi(u_n))_{n\ge1}$ is bounded in $\mathbb{R}$ and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

This is a compactness-type condition on the functional φ to offset the fact that X is in general infinite dimensional, hence not locally compact. It leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ. A central result of this theory is the so-called mountain pass theorem due to Ambrosetti-Rabinowitz [2] which we recall here in a slightly more general form (see, for example, Gasiński-Papageorgiou [7, p. 648]).

Theorem 2.2. *Let $\varphi \in C^1(X)$ be a functional satisfying the C-condition and let $u_1, u_2 \in X, \|u_2 - u_1\|_X > \rho > 0$,*

$$\max\{\varphi(u_1), \varphi(u_2)\} < \inf\{\varphi(u) : \|u - u_1\|_X = \rho\} =: m_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_1, \gamma(1) = u_2\}$. Then $c \ge m_\rho$ with c being a critical value of φ.

By $H^1(\Omega)$ we denote the usual Hilbert space with inner product

$$(u,h) = \int_\Omega uh dx + \int_\Omega (\nabla u, \nabla h)_{\mathbb{R}^N} dx \quad \text{for all } u,h \in H^1(\Omega)$$

and corresponding norm

$$\|u\| = \left[\|u\|_2^2 + \|\nabla u\|_2^2\right]^{\frac{1}{2}} \quad \text{for all } u \in H^1(\Omega),$$

where $\|\cdot\|_2$ stands for the norm in the Lebesgue space $L^2(\Omega)$. The norm of $\mathbb{R}^N$ is denoted by $\|\cdot\|_{\mathbb{R}^N}$ and $(\cdot,\cdot)_{\mathbb{R}^N}$ stands for the inner product in $\mathbb{R}^N$. For $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for $u \in H^1(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. It is well known that

$$u^\pm \in H^1(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

The Lebesgue measure on $\mathbb{R}^N$ is denoted by $|\cdot|_N$ and for a measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we define the Nemytskij operator corresponding to the function h by

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in H^1(\Omega).$$

Evidently, $x \mapsto N_h(u)(x)$ is measurable.

In addition to the Sobolev space $H^1(\Omega)$ we will also use the ordered Banach space $C^1(\overline{\Omega})$ and its positive cone

$$C^1(\overline{\Omega})_+ = \left\{u \in C^1(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior in $C(\overline{\Omega})$ given by

$$\operatorname{int}\left(C^1(\overline{\Omega})_+\right) = \left\{u \in C^1(\overline{\Omega})_+ : u(x) > 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

On the boundary $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Having this measure, we can define in the usual way the boundary Lebesgue spaces $L^s(\partial\Omega)$ for $1 \leq s \leq \infty$. From the theory of Sobolev spaces we know that there exists a unique linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ known as the trace map such that

$$\gamma_0(u) = u\big|_{\partial\Omega} \quad \text{for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

The trace map prescribes boundary values to Sobolev functions. Furthermore we know that the trace map is compact into $L^s(\partial\Omega)$ for every $s \in [1, 2_*)$, where 2_* is the critical exponent of 2 given by

$$2_* = \begin{cases} \frac{2(N-1)}{N-2} & \text{if } N \geq 3, \\ \infty & \text{if } N = 1,2. \end{cases}$$

Moreover it holds

$$\operatorname{im}\gamma_0 = H^{\frac{1}{2},2}(\partial\Omega) \quad \text{and} \quad \ker\gamma_0 = H_0^1(\Omega).$$

From now on, for the sake of notational simplicity, we drop the usage of the trace operator γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

As we already described in the Introduction we will use the spectrum of the differential operator $u \to -\Delta u + \xi(x)u$ with Robin boundary condition. So, we consider the following linear eigenvalue problem

$$\begin{aligned} -\Delta u + \xi(x)u &= \hat{\lambda} u && \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(x)u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

In this eigenvalue problem we impose the following conditions on its data:

- $\xi \in L^{\frac{N}{2}}(\Omega)$ if $N \geq 3$, $\xi \in L^s(\Omega)$ with $s \in (1, +\infty)$ if $N = 2$ and $\xi \in L^1(\Omega)$ if $N = 1$.
- $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(x) \geq 0$ for all $x \in \partial\Omega$.

Let $\gamma : H^1(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\gamma(u) = \|\nabla u\|_2^2 + \int_\Omega \xi(x)u^2 dx + \int_{\partial\Omega} \beta(x)u^2 d\sigma.$$

From D'Aguì-Marano-Papageorgiou [6] we know that there exist $\mu, c_0 > 0$ such that

$$\gamma(u) + \mu\|u\|_2^2 \geq c_0\|u\|^2 \quad \text{for all } u \in H^1(\Omega). \tag{2.2}$$

Then using (2.2) and the spectral theorem for compact self-adjoint operators on a Hilbert space, we can have a complete description of the spectrum of (2.1). This consists of a strictly increasing sequence $\left(\hat{\lambda}_k\right)_{k\in\mathbb{N}}$ of eigenvalues such that $\hat{\lambda}_k \to +\infty$ as $k \to +\infty$. By $E\left(\hat{\lambda}_k\right), k \in \mathbb{N}$, we denote the corresponding eigenspace. These are finite dimensional subspaces of $H^1(\Omega)$ and we have the following orthogonal direct sum decomposition

$$H^1(\Omega) = \overline{\bigoplus_{k\geq 1} E\left(\hat{\lambda}_k\right)}.$$

The eigenvalues of (2.1) have the following properties:

- $\hat{\lambda}_1$ is simple, that is $\dim E(\hat{\lambda}_1) = 1$.
- $$\hat{\lambda}_1 = \inf\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in H^1(\Omega), u \neq 0\right]. \tag{2.3}$$
- For $m \geq 2$ we have

$$\begin{aligned} \hat{\lambda}_m &= \inf\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in \overline{\bigoplus_{k\geq m} E\left(\hat{\lambda}_k\right)}, u \neq 0\right] \\ &= \sup\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in \bigoplus_{k=1}^{m} E\left(\hat{\lambda}_k\right), u \neq 0\right]. \end{aligned} \tag{2.4}$$

The infimum in (2.3) is attained on $E(\hat{\lambda}_1)$ while both the infimum and the supremum in (2.4) are attained on $E(\hat{\lambda}_m)$. Evidently the elements of $E(\hat{\lambda}_1)$ have fixed sign while the elements of $E(\hat{\lambda}_m), m \geq 2$, are nodal, that is, sign changing. By $\hat{u}_1$ we denote the L^2-normalized (that is, $\|\hat{u}_1\|_2 = 1$) positive eigenfunction corresponding to $\hat{\lambda}_1$. If $\xi \in L^q(\Omega)$ with $q > N$, then the regularity theory of Wang [18] (see Lemmata 5.1 and 5.2) implies that $\hat{u}_1 \in C^1(\overline{\Omega})_+ \setminus \{0\}$. Moreover, Harnack's inequality (see, for example, Motreanu-Motreanu-Papageorgiou [12, p. 212]) we have $\hat{u}_1(x) > 0$ for all $x \in \overline{\Omega}$. If $\xi^+ \in L^\infty(\Omega)$, then the strong maximum principle (see, for example, Gasiński-Papageorgiou [7, p. 738]) implies that $\hat{u}_1 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. If $\xi \in L^q(\Omega)$

with $q > \frac{N}{2}$, then the eigenspaces $E(\hat{\lambda}_k), k \in \mathbb{N}$ have the so-called "Unique Continuation Property" (ucp for short) which says that if $u \in E(\hat{\lambda}_k)$ and u vanishes on a set of positive Lebesgue measure, then $u \equiv 0$.

The above properties lead to the following useful inequalities which can be found in D'Aguì-Marano-Papageorgiou [6].

Lemma 2.3.

(a) If $m \in \mathbb{N}$, $\vartheta \in L^\infty(\Omega)$ and $\vartheta(x) \leq \hat{\lambda}_m$ for a.a. $x \in \Omega, \vartheta \neq \hat{\lambda}_m$, then there exists a number $c_1 > 0$ such that

$$\gamma(u) - \int_\Omega \vartheta(x) u^2 dx \geq c_1 \|u\|^2 \quad \text{for all } u \in \overline{\bigoplus_{k \geq m} E\left(\hat{\lambda}_k\right)}.$$

(b) If $m \in \mathbb{N}$, $\eta \in L^\infty(\Omega)$ and $\eta(x) \geq \hat{\lambda}_m$ for a.a. $x \in \Omega, \eta \neq \hat{\lambda}_m$, then there exists a number $c_2 > 0$ such that

$$\gamma(u) - \int_\Omega \eta(x) u^2 dx \leq -c_2 \|u\|^2 \quad \text{for all } u \in \bigoplus_{k=1}^{m} E\left(\hat{\lambda}_k\right).$$

We will also consider the weighted version of the linear eigenvalue problem (2.1). So, let $\eta \in L^\infty(\Omega), \eta \geq 0, \eta \neq 0$. We consider the following linear eigenvalue problem

$$\begin{aligned} -\Delta u + \xi(x)u &= \hat{\lambda}\eta(x)u && \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(x)u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.5}$$

In the same way as for problem (2.1) we show that the linear eigenvalue problem (2.5) admits a strictly increasing sequence of eigenvalues $\left(\tilde{\lambda}_k(\eta)\right)_{k \in \mathbb{N}}$ such that $\tilde{\lambda}_k(\eta) \to +\infty$ as $k \to +\infty$. In this case, in the variational characterization of the eigenvalues, the Rayleigh quotient has the form

$$R(u) = \frac{\gamma(u)}{\int_\Omega \eta(x) u^2 dx} \quad \text{for all } u \in H^1(\Omega) \setminus \{0\}.$$

As a consequence of the ucp of the eigenspaces we infer the following strict monotonicity property of the map $\eta \to \tilde{\lambda}_m(\eta), m \in \mathbb{N}$.

Lemma 2.4. *If $\eta_1, \eta_2 \in L^\infty(\Omega), 0 \leq \eta_1 \leq \eta_2, \eta_1 \neq 0$ and $\eta_1 \neq \eta_2$, then, for all $m \in \mathbb{N}$ we have $\tilde{\lambda}_m(\eta_2) < \tilde{\lambda}_m(\eta_1)$.*

Next, let us recall some basic definitions and facts about Morse theory which will need in the sequel. Let X be a Banach space and let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$ the term $H_k(Y_1, Y_2)$ stands for the $k^{\underline{th}}$-relative singular homology group with integer coefficients. Recall that

$$H_k(Y_1, Y_2) = Z_k(Y_1, Y_2) \big/ B_k(Y_1, Y_2) \quad \text{for all } k \in \mathbb{N}_0,$$

where $Z_k(Y_1, Y_2)$ is the group of relative singular k-cycles of Y_1 mod Y_2 (that is, $Z_k(Y_1, Y_2) = \ker \partial_k$ with ∂_k being the boundary homomorphism) and $B_k(Y_1, Y_2)$ is the group of relative singular k-boundaries of Y_1 mod Y_2 (that is, $B_k(Y_1, Y_2) = \operatorname{im} \partial_{k+1}$). We know that $\partial_{k-1} \circ \partial_k = 0$ for all $k \in \mathbb{N}$, hence $B_k(Y_1, Y_2) \subseteq Z_k(Y_1, Y_2)$ and so the quotient

$$Z_k(Y_1, Y_2) \big/ B_k(Y_1, Y_2)$$

makes sense.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$
\begin{aligned}
\varphi^c &= \{u \in X : \varphi(u) \leq c\} && \text{(the sublevel set of } \varphi \text{ at } c),\\
K_\varphi &= \{u \in X : \varphi'(u) = 0\} && \text{(the critical set of } \varphi),\\
K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\} && \text{(the critical set of } \varphi \text{ at the level } c).
\end{aligned}
$$

For every isolated critical point $u \in K_\varphi^c$ the critical groups of φ at $u \in K_\varphi^c$ are defined by

$$
C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \geq 0,
$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology theory implies that the definition of critical groups above is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and that $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$
C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \geq 0.
$$

This definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. This is a consequence of the second deformation theorem (see, for example, Gasiński-Papageorgiou [7, p. 628]).

We now assume that K_φ is finite and introduce the following series in $t \in \mathbb{R}$:

$$
\begin{aligned}
M(t, u) &= \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, u) t^k \quad \text{for all } u \in K_\varphi,\\
P(t, \infty) &= \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, \infty) t^k.
\end{aligned}
$$

The Morse relation says that

$$
\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{2.6}
$$

with $Q(t) = \sum_{k \geq 0} \beta_k t^k$ being a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

By $A \in \mathcal{L}(H^1(\Omega), \left(H^1(\Omega)\right)^*)$ we denote the linear operator defined by

$$
\langle A(u), h \rangle = \int_\Omega (\nabla u, \nabla h)_{\mathbb{R}^N} \, dx \quad \text{for all } u, h \in H^1(\Omega).
$$

Furthermore, we say that a Banach space X has the Kadec-Klee property, if the following implication is true:

$$
u_n \rightharpoonup u \text{ in } X \text{ and } \|u_n\| \to \|u\| \quad \Longrightarrow \quad u_n \to u \text{ in } X.
$$

Locally uniformly convex Banach spaces, in particular Hilbert spaces, have the Kadec-Klee property. Finally, let

$$
m_0 = \min \left\{ k \in \mathbb{N} : \hat{\lambda}_k \geq 0 \right\}, \tag{2.7}
$$

that is, $\hat{\lambda}_{m_0}$ is the first nonnegative eigenvalue of (2.1). If $\xi \geq 0$ and $\beta \geq 0$, then $\hat{\lambda}_1 \geq 0$ and so $m_0 = 1$. Moreover, if $\xi \geq 0, \beta \geq 0$ and one of the two is different from zero, then $\hat{\lambda}_1 > 0$.

3. **Multiplicity Theorem.** In this section we prove a multiplicity theorem about the existence of three nontrivial solutions to problem (1.1) under conditions of double resonance.

Our hypotheses on the data of problem (1.1) are the following.

H(ξ): $\xi \in L^q(\Omega)$ with $q > N$ when $N \geq 2$ and $q = 1$ when $N = 1$; in addition $\xi^+ \in L^\infty(\Omega)$.

H(β): $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(x) \geq 0$ for all $x \in \partial\Omega$.

H(f): $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x, 0) = 0$ for a.a. $x \in \Omega$ and

(i) for every $\rho > 0$ there exists $a_\rho \in L^\infty(\Omega)$ such that

$$|f(x, s)| \leq a_\rho(x) \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R};$$

(ii) there exists $k \in \mathbb{N}$ with $k \geq \max\{m_0, 2\}$ such that

$$\hat{\lambda}_k \leq \liminf_{s\to\pm\infty} \frac{f(x, s)}{s} \leq \limsup_{s\to\pm\infty} \frac{f(x, s)}{s} \leq \hat{\lambda}_{k+1}$$

uniformly for a.a. $x \in \Omega$ where m_0 is given in (2.7);

(iii) if $F(x, s) = \int_0^s f(x, t)dt$, then

$$\lim_{s\to\pm\infty} [f(x, s)s - 2F(x, s)] = +\infty$$

uniformly for a.a. $x \in \Omega$;

(iv) there exist a function $\vartheta \in L^\infty(\Omega)$ and $c_3 > 0$ such that

$$\vartheta(x) \leq \hat{\lambda}_1 \quad \text{for a.a. } x \in \Omega, \quad \vartheta \neq \hat{\lambda}_1$$

and

$$-c_3 \leq \liminf_{s\to 0} \frac{f(x, s)}{s} \leq \limsup_{s\to 0} \frac{f(x, s)}{s} \leq \vartheta(x)$$

uniformly for a.a. $x \in \Omega$.

Now let $\mu > 0$ as in (2.2) and consider the following truncation-perturbation of the nonlinearity $f(x, \cdot)$

$$\hat{f}_+(x, s) = \begin{cases} 0, & \text{if } s \leq 0, \\ f(x, s) + \mu s, & \text{if } s > 0, \end{cases} \qquad \hat{f}_-(x, s) = \begin{cases} f(x, s) + \mu s, & \text{if } s < 0, \\ 0, & \text{if } s \geq 0. \end{cases} \tag{3.1}$$

It is clear that both functions are of Carathéodory type. We set $\hat{F}_\pm(x, s) = \int_0^s \hat{f}_\pm(x, t)dt$ and introduce the C^1-functionals $\hat{\varphi}_\pm : H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\pm(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u\|_2^2 - \int_\Omega \hat{F}_\pm(x, u)dx.$$

Furthermore, let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy (Euler) functional of problem (1.1) which is defined by

$$\varphi(u) = \frac{1}{2}\gamma(u) - \int_\Omega F(x, u)dx.$$

Evidently $\varphi \in C^1(H^1(\Omega))$.

Let us consider first the truncation functionals $\hat{\varphi}_\pm$.

Proposition 3.1. *If hypotheses H(ξ), H(β) and H(f) are satisfied, then the functionals $\hat{\varphi}_\pm$ fulfill the C-condition.*

Proof. We will show the assertion only for $\hat{\varphi}_+$, the proof for $\hat{\varphi}_-$ is similar. So, let $(u_n)_{n\geq 1} \subseteq H^1(\Omega)$ be a sequence such that

$$|\hat{\varphi}_+(u_n)| \leq M_1 \quad \text{for all } n \geq 1$$

with some $M_1 > 0$ and

$$(1 + \|u_n\|)\, \hat{\varphi}_+'(u_n) \to 0 \quad \text{in } \left(H^1(\Omega)\right)^*. \tag{3.2}$$

Due to (3.2) we get

$$\begin{aligned} &\left| \langle A(u_n), h\rangle + \int_\Omega (\xi(x) + \mu) u_n h dx + \int_{\partial\Omega} \beta(x) u_n h d\sigma - \int_\Omega \hat{f}_+(x, u_n) h dx \right| \\ &\leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \end{aligned} \tag{3.3}$$

for all $h \in H^1(\Omega)$ with $\varepsilon_n \to 0^+$. We choose $h = -u_n^- \in H^1(\Omega)$ in (3.3) to obtain

$$\gamma(u_n^-) + \mu \|u_n^-\|_2^2 \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}$$

due to the truncation (3.1) which implies $c_0\|u_n^-\|^2 \leq \varepsilon_n$ for all $n \in \mathbb{N}$ because of (2.2). This finally gives

$$u_n^- \to 0 \quad \text{in } H^1(\Omega). \tag{3.4}$$

We claim now that $(u_n)_{n\geq 1} \subseteq H^1(\Omega)$ is bounded. Arguing by contradiction, suppose that by passing to a subsequence if necessary, we have

$$\|u_n^+\| \to \infty. \tag{3.5}$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$ for all $n \geq 1$. Then $\|y_n\| = 1, y_n \geq 0$ for all $n \geq 1$ and so, by the Sobolev embedding theorem and the compactness of the trace map, we may assume that

$$y_n \rightharpoonup y \quad \text{in } H^1(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^{\frac{2q}{q-1}}(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.6}$$

Combining (3.1), (3.3) and (3.4) results in

$$\left| \langle A(u_n^+), h\rangle + \int_\Omega \xi(x) u_n^+ h dx + \int_{\partial\Omega} \beta(x) u_n^+ h d\sigma - \int_\Omega \hat{f}_+(x, u_n^+) h dx \right| \leq \varepsilon_n' \|h\|$$

for all $h \in H^1(\Omega)$ with $\varepsilon_n' \to 0^+$. Therefore, we have

$$\begin{aligned} &\left| \langle A(y_n), h\rangle + \int_\Omega \xi(x) y_n h dx + \int_{\partial\Omega} \beta(x) y_n h d\sigma - \int_\Omega \frac{N_f(u_n^+)}{\|u_n^+\|} h dx \right| \\ &\leq \frac{\varepsilon_n'}{\|u_n^+\|} \|h\| \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{3.7}$$

Hypotheses H(f)(i), (ii) imply that

$$|f(x, s)| \leq c_4(1 + |s|) \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R} \text{ with } c_4 > 0.$$

This fact along with (3.5) ensures

$$\left(\frac{N_f(u_n^+)}{\|u_n^+\|} \right)_{n\geq 1} \subseteq L^2(\Omega) \text{ is bounded.}$$

So, by passing to a subsequence if necessary and by applying condition H(f)(ii), we obtain

$$\frac{N_f(u_n^+)}{\|u_n^+\|} \rightharpoonup \eta(x) y \quad \text{in } L^2(\Omega) \text{ as } n \to \infty \tag{3.8}$$

with $\hat{\lambda}_k \leq \eta(x) \leq \hat{\lambda}_{k+1}$ for a.a. $x \in \Omega$, see the proof of Proposition 16 in Aizicovici-Papageorgiou-Staicu [1]. Now let us pass to the limit in (3.7) as $n \to \infty$ while we use (3.6) and (3.8). Thus

$$\langle A(y), h\rangle + \int_\Omega \xi(x) y h dx + \int_{\partial\Omega} \beta(x) y h d\sigma = \int_\Omega \eta(x) y h dx$$

for all $h \in H^1(\Omega)$. This gives (see Papageorgiou-Rădulescu [14])

$$\begin{aligned} -\Delta y + \xi(x) y &= \eta(x) y \qquad \text{in } \Omega,\\ \frac{\partial y}{\partial n} + \beta(x) y &= 0 \qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{3.9}$$

Now we choose $h = y_n - y \in H^1(\Omega)$ in (3.7), pass to the limit and use (3.6) as well as (3.8). Then

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0,$$

hence $\|\nabla y_n\|_2 \to \|\nabla y\|_2$ which by the Kadec-Klee property in combination with (3.6) finally gives $y_n \to y$ in $H^1(\Omega)$. Thus

$$\|y\| = 1 \text{ and } y \geq 0. \tag{3.10}$$

First suppose that

$$\eta(x) = \hat{\lambda}_k \quad \text{or} \quad \eta(x) = \hat{\lambda}_{k+1} \quad \text{for a.a. } x \in \Omega,$$

see (3.8). Because of (3.9) and (3.10) and since $k \geq 2$ we infer that y is nodal which contradicts (3.10).

So, assume that $\eta \neq \hat{\lambda}_k$ and $\eta \neq \hat{\lambda}_{k+1}$. Applying Lemma 2.4 implies that

$$\tilde{\lambda}_k(\eta) < \tilde{\lambda}_k(\hat{\lambda}_k) = 1 \quad \text{and} \quad 1 = \tilde{\lambda}_{k+1}(\hat{\lambda}_{k+1}) < \tilde{\lambda}_{k+1}(\eta),$$

which gives $y = 0$ (see (3.9)) and so a contradiction because of (3.10).

This proves that

$$\left(u_n^+\right)_{n\geq 1} \subseteq H^1(\Omega) \text{ is bounded.} \tag{3.11}$$

Due to (3.4) and (3.11), we get then

$$(u_n)_{n\geq 1} \subseteq H^1(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \rightharpoonup u \quad \text{in } H^1(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{\frac{2q}{q-1}}(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.12}$$

Taking $h = u_n - u \in H^1(\Omega)$ in (3.3), passing to the limit as $n \to \infty$ and applying (3.12) yields

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0.$$

This shows that $\|\nabla u_n\|_2 \to \|\nabla u\|_2$ and taking the Kadec-Klee property along with (3.12) into account this finally implies that $u_n \to u$ in $H^1(\Omega)$. This proves that the functional $\hat{\varphi}_+$ satisfies the C-condition.

Similarly we show that the functional $\hat{\varphi}_-$ satisfies the C-condition. $\square$

Let us now prove a similar result for the energy functional $\varphi : H^1(\Omega) \to \mathbb{R}$.

Proposition 3.2. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ are satisfied, then the functional φ fulfills the C-condition.*

Proof. Consider a sequence $(u_n)_{n\geq 1} \subseteq H^1(\Omega)$ such that

$$|\varphi(u_n)| \leq M_2 \quad \text{for some } M_2 > 0 \text{ and for all } n \geq 1, \tag{3.13}$$

$$(1 + \|u_n\|)\, \varphi'(u_n) \to 0 \quad \text{in } \left(H^1(\Omega)\right)^* \text{ as } n \to \infty. \tag{3.14}$$

Assertion (3.14) implies

$$\begin{aligned} &\left| \langle A(u_n), h\rangle + \int_\Omega \xi(x) u_n h dx + \int_{\partial\Omega} \beta(x) u_n h d\sigma - \int_\Omega f(x, u_n) h dx \right| \\ &\leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all } h \in H^1(\Omega) \text{ with } \varepsilon_n \to 0^+. \end{aligned} \tag{3.15}$$

Taking $h = u_n \in H^1(\Omega)$ in (3.15) gives

$$-\gamma(u_n) + \int_\Omega f(x, u_n) u_n dx \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{3.16}$$

On the other hand, by using (3.13), we have

$$\gamma(u_n) - \int_\Omega 2F(x, u_n) dx \leq 2M_2 \quad \text{for all } n \in \mathbb{N}. \tag{3.17}$$

Adding (3.16) and (3.17) we obtain

$$\int_\Omega \left[f(x, u_n) u_n - 2F(x, u_n)\right] dx \leq M_3 \tag{3.18}$$

for some $M_3 > 0$ and for all $n \in \mathbb{N}$. We are going to show that the sequence $(u_n)_{n\geq 1} \subseteq H^1(\Omega)$ is bounded. Arguing indirectly, suppose that at least for a subsequence, we have

$$\|u_n\| \to +\infty.$$

Let $y_n = \frac{u_n}{\|u_n\|}$ for all $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \rightharpoonup y \quad \text{in } H^1(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^{\frac{2q}{q-1}}(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.19}$$

Inequality (3.15) can be rewritten as follows

$$\begin{aligned} &\left| \langle A(y_n), h\rangle + \int_\Omega \xi(x) y_n h dx + \int_{\partial\Omega} \beta(x) y_n h d\sigma - \int_\Omega \frac{N_f(u_n)}{\|u_n\|} h dx \right| \\ &\leq \frac{\varepsilon_n \|h\|}{(1 + \|u_n\|)\|u_n\|} \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{3.20}$$

Recall that with the aid of hypotheses H(f)(ii) one can show that

$$\frac{N_f(u_n)}{\|u_n\|} \rightharpoonup \eta y \text{ in } L^2(\Omega) \text{ as } n \to \infty \tag{3.21}$$

with $\hat{\lambda}_k \leq \eta(x) \leq \hat{\lambda}_{k+1}$ for a.a. $x \in \Omega$. Taking $h = y_n - y \in H^1(\Omega)$ in (3.20), passing to the limit as $n \to \infty$ and using (3.19) as well as (3.21) gives

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0,$$

which directly implies that

$$y_n \to y \text{ in } H^1(\Omega) \text{ and } \|y\| = 1, \tag{3.22}$$

as before because of the Kadec-Klee property. Now, passing to the limit in (3.20) and applying (3.20) and (3.21) leads to

$$\langle A(y), h\rangle + \int_\Omega \xi(x) y h dx + \int_{\partial\Omega} \beta(x) y h d\sigma = \int_\Omega \eta(x) y h dx$$

for all $h \in H^1(\Omega)$. As before, see Papageorgiou-Rădulescu [14], this is equivalent to

$$\begin{aligned} -\Delta y + \xi(x) y &= \eta(x) y \qquad \text{in } \Omega, \\ \frac{\partial y}{\partial n} + \beta(x) y &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.23}$$

First suppose that

$$\eta \neq \hat{\lambda}_k \quad \text{and} \quad \eta \neq \hat{\lambda}_{k+1}.$$

Then, (3.21) and Lemma 2.4 implies that

$$\tilde{\lambda}_k(\eta) < \tilde{\lambda}_k(\hat{\lambda}_k) = 1 \quad \text{and} \quad 1 = \tilde{\lambda}_{k+1}(\hat{\lambda}_{k+1}) < \tilde{\lambda}_{k+1}(\eta),$$

which gives $y = 0$ (see (3.23)) and so contradicts (3.22).

Next suppose that

$$\eta(x) = \hat{\lambda}_k \quad \text{or} \quad \eta(x) = \hat{\lambda}_{k+1} \quad \text{for a.a. } x \in \Omega,$$

Then, (3.23) and the ucp of the eigenspaces (see Section 2) imply $y(x) \neq 0$ for a.a. $x \in \Omega$. Hence,

$$|u_n(x)| \to +\infty \quad \text{for a.a. } x \in \Omega \text{ as } n \to \infty. \tag{3.24}$$

From (3.24), hypotheses H(f)(iii) and Fatou's Lemma we obtain

$$\int_\Omega [f(x, u_n) u_n - 2F(x, u_n)]\, dx \to +\infty,$$

which contradicts (3.18).

This proves that $(u_n)_{n\geq 1} \subseteq H^1(\Omega)$ is bounded. So we may assume that

$$u_n \rightharpoonup u \quad \text{in } H^1(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{\frac{2q}{q-1}}(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.25}$$

Taking $h = u_n - u \in H^1(\Omega)$ in (3.15), passing to the limit as $n \to \infty$ and applying (3.25), gives

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0.$$

Hence, as before, $u_n \to u$ in $H^1(\Omega)$. Therefore we conclude that φ satisfies the C-condition. □

Proposition 3.3. *If hypotheses H(ξ), H(β) and H(f) are satisfied, then $u = 0$ is a local minimizer of the functionals $\hat{\varphi}_\pm$ and φ.*

Proof. As before we are going to show the assertion only for the functional $\hat{\varphi}_+$, the proofs for $\hat{\varphi}_-$ and for φ are very similar.

Hypotheses H(f)(i), (ii) imply that

$$|F(x, s)| \leq c_5(1 + s^2) \tag{3.26}$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for some $c_5 > 0$. Let $r > 2$ and $\varepsilon > 0$. Then (3.26) along with hypotheses H(f)(iv) imply that we can find $c_6 = c_6(\varepsilon) > 0$ such that

$$|F(x, s)| \leq \frac{1}{2}\left(\vartheta(x) + \varepsilon\right) s^2 + c_6 |s|^r \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R}. \tag{3.27}$$

Applying now (3.1) and (3.27) we have for every $u \in H^1(\Omega)$

$$\begin{aligned}
\hat{\varphi}_+(u) &= \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u\|_2^2 - \int_\Omega \hat{F}_+(x,u)dx \\
&= \frac{1}{2}\gamma(u^-) + \frac{\mu}{2}\|u^-\|_2^2 + \frac{1}{2}\gamma(u^+) - \int_\Omega F(x,u^+)dx \\
&\geq \frac{c_0}{2}\|u^-\|^2 + \frac{1}{2}\left[\gamma(u^+) - \int_\Omega \vartheta(x)(u^+)^2dx\right] - \frac{\varepsilon}{2}\|u^+\|^2 - c_7\|u\|^r \\
&\geq \frac{c_0}{2}\|u^-\|^2 + \frac{1}{2}(c_8-\varepsilon)\|u^+\|^2 - c_7\|u\|^r
\end{aligned} \tag{3.28}$$

for some $c_7, c_8 > 0$. Choosing $0 < \varepsilon < c_8$ we see from (3.28) that

$$\hat{\varphi}_+(u) \geq c_9\|u\|^2 - c_7\|u\|^r \quad \text{for some } c_9 > 0.$$

Because $r > 2$ we find $\rho \in (0,1)$ small enough such that

$$\hat{\varphi}_+(0) = 0 < \hat{\varphi}_+(u) \text{ for all } u \in H^1(\Omega) \text{ with } 0 < \|u\| < \rho.$$

That means $u = 0$ is a strict local minimizer of $\hat{\varphi}_+$.

Similarly we show that $u = 0$ is a strict local minimizer for $\hat{\varphi}_-$ and φ. □

Proposition 3.4. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ are satisfied, then*

$$K_{\hat{\varphi}_+} \subseteq \operatorname{int}\left(C^1(\overline{\Omega})_+\right) \cup \{0\} \quad \textit{and} \quad K_{\hat{\varphi}_-} \subseteq -\operatorname{int}\left(C^1(\overline{\Omega})_+\right) \cup \{0\}$$

Proof. Let $u \in K_{\hat{\varphi}_+}$ with $u \neq 0$, that is,

$$\langle A(u), h\rangle + \int_\Omega (\xi(x)+\mu)uhdx + \int_{\partial\Omega} \beta(x)uhd\sigma = \int_\Omega \hat{f}_+(x,u)hdx \tag{3.29}$$

for all $h \in H^1(\Omega)$. Taking $h = -u^- \in H^1(\Omega)$ in (3.29) gives

$$\gamma(u^-) + \mu\|u^-\|_2^2 = 0,$$

due to the truncations in (3.1). Using (2.2) implies $c_0\|u^-\|^2 \leq 0$ and so $u \geq 0, u \neq 0$. Hence (3.29) becomes

$$\langle A(u), h\rangle + \int_\Omega \xi(x)uhdx + \int_{\partial\Omega} \beta(x)uhd\sigma = \int_\Omega f(x,u)hdx$$

for all $h \in H^1(\Omega)$ which means that

$$\begin{aligned}
-\Delta u + \xi(x)u &= f(x,u) && \text{in } \Omega, \\
\frac{\partial u}{\partial n} + \beta(x)u &= 0 && \text{on } \partial\Omega.
\end{aligned} \tag{3.30}$$

Hypotheses H(f)(i), (ii), (iv) imply that

$$|f(x,s)| \leq c_{10}|s| \tag{3.31}$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for some $c_{10} > 0$. Now we introduce the function

$$k(x) = \begin{cases} \frac{f(x,u(x))}{u(x)} & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0, \end{cases}$$

which belongs to $L^\infty(\Omega)$ because of (3.31). With the representation of k, we can rewrite (3.30) as follows

$$\begin{aligned}
-\Delta u &= (k(x) - \xi(x))u(x) && \text{in } \Omega, \\
\frac{\partial u}{\partial n} + \beta(x)u &= 0 && \text{on } \partial\Omega.
\end{aligned} \tag{3.32}$$

Then (3.32) and Lemma 5.1. of Wang [18] imply that

$$u \in L^{\infty}(\Omega).$$

Since hypotheses H(ξ) and (3.32) it follows that

$$\Delta u \in L^{q}(\Omega).$$

Applying the Calderon-Zygmund estimates (see Wang [18, Lemma 5.2]) we obtain that

$$u \in W^{2,q}(\Omega).$$

So, by the Sobolev embedding theorem we have $W^{2,q}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$ with $\alpha = 1 - \frac{N}{q} > 0$. Therefore

$$u \in C^{1}(\overline{\Omega})_{+} \setminus \{0\}.$$

From (3.31) it is clear that, for every $\rho > 0$, we can find $\hat{\xi}_\rho > 0$ such that

$$f(x,s) + \hat{\xi}_\rho s \geq 0 \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in [0,\rho]. \tag{3.33}$$

Using (3.30) and (3.33) with $\rho = \|u\|_\infty$ gives

$$\Delta u(x) \leq \left(\|\xi^{+}\|_\infty + \hat{\xi}_\rho \right) u(x) \quad \text{for a.a. } x \in \Omega,$$

due to hypothesis H(ξ). Applying the strong maximum principle we obtain $u \in \operatorname{int}\left(C^{1}(\overline{\Omega})_{+}\right)$, see, for instance, Gasiński-Papageorgiou [7, p. 738].

Similarly we can show that

$$K_{\hat{\varphi}_-} \subseteq -\operatorname{int}\left(C^{1}(\overline{\Omega})_{+}\right) \cup \{0\}.$$

□

Now we are ready to prove the existence of two nontrivial constant sign solutions of problem (1.1).

Proposition 3.5. *If hypotheses H(ξ), H(β) and H(f) are satisfied, then problem* (1.1) *admits two nontrivial constant sign solutions*

$$u_0 \in \operatorname{int}\left(C^{1}(\overline{\Omega})_{+}\right) \quad \text{and} \quad v_0 \in -\operatorname{int}\left(C^{1}(\overline{\Omega})_{+}\right).$$

Proof. Based on Proposition 3.4 we see that we can assume that the critical sets $K_{\hat{\varphi}_+}$ and $K_{\hat{\varphi}_-}$ are finite or otherwise we already have an infinity of positive and negative solutions of problem (1.1) and so we are done.

Proposition 3.3 implies the existence of $\rho \in (0,1)$ small enough such that

$$\hat{\varphi}_+(0) = 0 < \inf\left[\hat{\varphi}_+(u) : \|u\| = \rho\right] = \hat{m}_+, \tag{3.34}$$

see Aizicovici-Papageorgiou-Staicu[1, Proof of Proposition 29]. Hypothesis H(f)(ii) implies that

$$\hat{\varphi}_+\left(t\hat{u}_1\right) \to -\infty \quad \text{as } t \to +\infty \tag{3.35}$$

because of $k \geq 2$ and $\hat{u}_1 \in \operatorname{int}\left(C^{1}(\overline{\Omega})_{+}\right)$. Finally, Proposition 3.1 states

$$\hat{\varphi}_+ \text{ satisfies the } C\text{-condition.} \tag{3.36}$$

Hence, (3.34), (3.35) and (3.36) allow the usage of the mountain pass theorem stated in Theorem 2.2. So we find $u_0 \in H^1(\Omega)$ such that

$$u_0 \in K_{\hat{\varphi}_+} \quad \text{and} \quad \hat{m}_+ \leq \hat{\varphi}_+(u_0). \tag{3.37}$$

From (3.34), (3.37) and Proposition 3.4 it follows that $u_0 \in \operatorname{int}\left(C^{1}(\overline{\Omega})_{+}\right)$.

Similarly, working with the functional $\hat{\varphi}_-$ instead, we prove the existence of a negative solution $v_0 \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. □

In order to prove the existence of a third smooth solution of problem (1.1) we will apply Morse theory in terms of critical groups.

Proposition 3.6. *If hypotheses H(ξ), H(β) and H(f) are satisfied, then*

$$C_m(\varphi,\infty)=\delta_{m,d_k}\mathbb{Z} \quad \text{for all } m\in\mathbb{N}_0 \text{ with } d_k=\dim\bigoplus_{i=1}^k E\left(\hat{\lambda}_i\right).$$

Proof. Let $\lambda \in \left(\hat{\lambda}_k, \hat{\lambda}_{k+1}\right)$ and consider the C^2-functional $\psi : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi(u)=\frac{1}{2}\gamma(u)-\frac{\lambda}{2}\|u\|_2^2.$$

The choice of $\lambda > 0$ implies that $K_\psi = \{0\}$ and $u = 0$ is nondegenerate. Hence we have

$$C_m(\psi,\infty)=C_m(\psi,0)=\delta_{m,d_k}\mathbb{Z} \quad \text{for all } m\in\mathbb{N}_0 \tag{3.38}$$

with $d_k = \dim \bigoplus_{i=1}^k E\left(\hat{\lambda}_i\right)$, see Motreanu-Motreanu-Papageorgiou [12, pp. 160 and 155].

Consider the homotopy $h : [0,1] \times H^1(\Omega) \to \mathbb{R}$ defined by

$$h(t,u)=(1-t)\varphi(u)+t\psi(u)$$

for all $t \in [0,1]$ and for all $u \in H^1(\Omega)$.

Claim. *We can find $\tau \in \mathbb{R}$ and $\varsigma > 0$ such that*

$$h(t,u)\le\tau \quad \Longrightarrow \quad (1+\|u\|)\left\|h_u'(t,u)\right\|_*\ge\varsigma$$

for all $t \in [0,1]$ and for all $u \in H^1(\Omega)$.

Suppose that the Claim is not true. Since h is bounded on bounded sets we can find $(t_n)_{n\ge1} \subseteq [0,1]$ and $(u_n)_{n\ge1} \subseteq H^1(\Omega)$ such that

$$\begin{aligned}
&t_n\to t, \quad \|u_n\|\to+\infty, \quad h(t_n,u_n)\to-\infty \quad \text{and}\\
&(1+\|u_n\|)h_u'(t_n,u_n)\to 0 \quad \text{in } (H^1(\Omega))^*.
\end{aligned} \tag{3.39}$$

From the last convergence in (3.39) we have

$$\begin{aligned}
&|\langle A(u_n),h\rangle|+\int_\Omega \xi(x)u_nhdx\int_{\partial\Omega}\beta(x)u_nhd\sigma\\
&-(1-t_n)\int_\Omega f(x,u_n)hdx-t_n\int_\Omega \lambda u_nhdx\le\frac{\varepsilon_n\|h\|}{1+\|u_n\|}
\end{aligned} \tag{3.40}$$

for all $h \in H^1(\Omega)$ with $\varepsilon_n \to 0^+$. Let $y_n = \frac{u_n}{\|u_n\|}$ for all $n \ge 1$. Then $\|y_n\| = 1$ for all $n \ge 1$ and so we may assume that

$$y_n\rightharpoonup y \quad \text{in } H^1(\Omega) \quad \text{and} \quad y_n\to y \quad \text{in } L^{\frac{2q}{q-1}}(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.41}$$

Rewriting (3.40) gives

$$\begin{aligned}
&|\langle A(y_n),h\rangle|+\int_\Omega \xi(x)y_nhdx\int_{\partial\Omega}\beta(x)y_nhd\sigma\\
&-(1-t_n)\int_\Omega \frac{N_f(u_n)}{\|u_n\|}hdx-t_n\int_\Omega \lambda y_nhdx\le\frac{\varepsilon_n\|h\|}{(1+\|u_n\|)\|u_n\|}
\end{aligned} \tag{3.42}$$

for all $n \in \mathbb{N}$. Recall that

$$\frac{N_f(u_n)}{\|u_n\|} \rightharpoonup \eta y \quad \text{in } L^2(\Omega) \quad \text{with} \quad \hat{\lambda}_k \leq \eta(x) \leq \hat{\lambda}_{k+1} \quad \text{for a.a. } x \in \Omega, \tag{3.43}$$

see hypothesis H(f)(ii) and the proof of Proposition 3.2. Now, choosing $h = y_n - y \in H^1(\Omega)$ in (3.42), passing to the limit as $n \to \infty$ and using (3.41) as well as (3.43) yields

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0,$$

which implies because of the Kadec-Klee property that $y_n \to y$. This gives

$$\|y\| = 1. \tag{3.44}$$

Passing to the limit in (3.42) as $n \to \infty$ and applying (3.43), we obtain

$$\langle A(y), h\rangle + \int_\Omega \xi(x) y h dx \int_{\partial\Omega} \beta(x) y h d\sigma = \int_\Omega \eta_t(x) y dx \tag{3.45}$$

for all $h \in H^1(\Omega)$ with $\eta_t(x) = (1-t)\eta(x) + t\lambda \in \left[\hat{\lambda}_k, \hat{\lambda}_{k+1}\right]$ for a.a. $x \in \Omega$. This means

$$\begin{aligned} -\Delta y + \xi(x) y &= \eta_t(x) y \qquad && \text{in } \Omega, \\ \frac{\partial y}{\partial n} + \beta(x) y &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.46}$$

Reasoning as in the proof of Proposition 3.2 by applying (3.44), (3.45), (3.46) and hypothesis H(f)(iii), we reach a contradiction. So, (3.39) cannot be true, hence the Claim holds.

As in the proof of Proposition 3.1 we can easily show that $h(t,\cdot)$ satisfies the C-condition for every $t \in [0,1]$. Therefore, Proposition 3.2 of Liang-Su [11] (see also Chang [5]) gives

$$C_m(h(0,\cdot),\infty) = C_m(h(1,\cdot),\infty) \quad \text{for all } m \in \mathbb{N}_0$$

and so

$$C_m(\varphi,\infty) = C_m(\psi,\infty) \quad \text{for all } m \in \mathbb{N}_0.$$

Becuase of (3.38) we derive

$$C_m(\varphi,\infty) = \delta_{m,d_k}\mathbb{Z} \quad \text{for all } m \in \mathbb{N}_0.$$

□

We also compute the critical groups at infinity of the functionals $\hat{\varphi}_\pm$. Recall that without any loss of generality, we can assume that the critical sets $K_{\hat{\varphi}_\pm}$ are finite.

Proposition 3.7. *If hypotheses H(ξ), H(β) and H(f) are satisfied, then*

$$C_m(\hat{\varphi}_\pm,\infty) = 0 \quad \textit{for all } m \in \mathbb{N}_0.$$

Proof. We will do the proof only for $\hat{\varphi}_+$, the proof for $\hat{\varphi}_-$ is very similar.

Let $\lambda \in \left(\hat{\lambda}_k, \hat{\lambda}_{k+1}\right)$ and consider the C^1-functional $\hat{\psi}_+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_+(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u^-\|_2^2 - \frac{\lambda}{2}\|u^+\|_2^2.$$

We introduce the homotopy $h_+ : [0,1] \times H^1(\Omega) \to \mathbb{R}$ defined by

$$h_+(t,u) = (1-t)\hat{\varphi}_+(u) + t\hat{\psi}_+(u).$$

Claim 1. *We can find* $\hat{\tau} \in \mathbb{R}$ *and* $\hat{\varsigma} > 0$ *such that*

$$\hat{h}_+(t,u) \leq \hat{\tau} \quad \Longrightarrow \quad (1+\|u\|)\left\|\left(\hat{h}_+\right)'_u\right\|_* \geq \hat{\varsigma} \quad \textit{for all } t \in [0,1].$$

Again we argue by contradiction. So suppose that we can find $(t_n)_{n\geq 1} \subseteq [0,1]$ and $(u_n)_{n\geq 1} \subseteq H^1(\Omega)$ such that

$$\begin{aligned} & t_n \to t, \quad \|u_n\| \to +\infty, \quad \hat{h}_+(t_n,u_n) \to -\infty \quad \text{and} \\ & (1+\|u_n\|)\left(\hat{h}_+\right)'_u(t_n,u_n) \to 0 \quad \text{in } (H^1(\Omega))^*, \end{aligned} \tag{3.47}$$

see the proof of Proposition 3.6. The last convergence in (3.47) gives

$$\begin{aligned} & \Bigg| \langle A(u_n),h\rangle + \int_\Omega \xi(x) u_n h dx \int_{\partial\Omega} \beta(x) u_n h d\sigma - \int_\Omega \mu u_n^- h dx \\ & -(1-t_n)\int_\Omega f(x,u_n^+) h dx - t_n \int_\Omega \lambda u_n^+ h dx \Bigg| \leq \frac{\varepsilon_n \|h\|}{1+\|u_n\|} \end{aligned} \tag{3.48}$$

for all $h \in H^1(\Omega)$ with $\varepsilon_n \to 0^+$. Choosing $h = -u_n^- \in H^1(\Omega)$ in (3.48) yields

$$\gamma(u_n^-) + \mu\|u_n^-\|_2^2 \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}$$

due to the truncations defined in (3.1). Then, thanks to (2.2) we derive $c_0\|u_n^-\|^2 \leq \varepsilon_n$ for all $n \in \mathbb{N}$ and so

$$u_n^- \to 0 \quad \text{in } H^1(\Omega). \tag{3.49}$$

From (3.47) and (3.49) it follows that

$$\|u_n^+\| \to \infty \text{ as } n \to \infty.$$

We set $y_n = \frac{u_n^+}{\|u_n^+\|}$ for all $n \geq 1$. Then $\|y_n\| = 1, y_n \geq 0$ for all $n \geq 1$ and so we may assume that

$$y_n \rightharpoonup y \quad \text{in } H^1(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^{\frac{2q}{q-1}}(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.50}$$

From (3.48) and (3.49) we have

$$\begin{aligned} & \Bigg| \langle A(y_n),h\rangle + \int_\Omega \xi(x) y_n h dx + \int_{\partial\Omega} \beta(x) y_n h d\sigma \\ & -(1-t_n)\int_\Omega \frac{N_f(u_n^+)}{\|u_n^+\|} h dx - t_n \int_\Omega \lambda y_n^+ h dx \Bigg| \leq \varepsilon_n'\|h\| \end{aligned} \tag{3.51}$$

for all $n \in \mathbb{N}$ with $\varepsilon_n' \to 0^+$. Recall that

$$\frac{N_f(u_n^+)}{\|u_n^+\|} \rightharpoonup \eta y \quad \text{in } L^2(\Omega) \quad \text{with} \quad \hat{\lambda}_k \leq \eta(x) \leq \hat{\lambda}_{k+1} \quad \text{for a.a. } x \in \Omega, \tag{3.52}$$

see (3.8). If we choose $h = y_n - y \in H^1(\Omega)$ in (3.51), pass to the limit as $n \to \infty$ and use (3.50) as well as (3.52), then

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0,$$

which means $\|\nabla y_n\|_2 \to \|\nabla y\|_2$. Because of the Kadec-Klee property and (3.50) we get $y_n \to y$ in $H^1(\Omega)$ and so

$$\|y\| = 1, \quad y \geq 0. \tag{3.53}$$

Passing to the limit in (3.51) as $n \to \infty$ and applying (3.52) we obtain

$$\langle A(y), h\rangle + \int_\Omega \xi(x)yhdx + \int_{\partial\Omega} \beta(x)yhd\sigma = \int_\Omega \eta_t(x)yhdx$$

for all $h \in H^1(\Omega)$ with $\eta_t(x) = (1-t)\eta(x) + t\lambda$. This means

$$\begin{aligned} -\Delta y + \xi(x)y &= \eta_t(x)y && \text{in } \Omega, \\ \frac{\partial y}{\partial n} + \beta(x)y &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.54}$$

Now, by applying (3.53) and (3.54) and following the ideas in the proof of Proposition 3.1, we reach a contradiction and this proves Claim 1.

A similar argument also shows that $\hat{h}_+(t,\cdot)$ satisfies the C-condition for all $t \in [0,1]$. So, Proposition 3.2 of Liang-Su [11] (see also Chang [5, Theorem 5.1.2 on page 334]) implies that

$$C_m\left(\hat{h}_+(0,\cdot),\infty\right) = C_m\left(\hat{h}_+(1,\cdot),\infty\right) \quad \text{for all } m \in \mathbb{N}_0,$$

which gives

$$C_m\left(\hat{\varphi}_+,\infty\right) = C_m\left(\hat{\psi}_+,\infty\right) \quad \text{for all } m \in \mathbb{N}_0. \tag{3.55}$$

Now, let us consider the homotopy $\hat{h}_+^* : [0,1] \times H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{h}_+^*(t,u) = \hat{\psi}_+(u) - t\int_\Omega udx.$$

Claim 2. $\left(\hat{h}_+^*\right)'_u(t,u) \neq 0$ *for all* $t \in [0,1]$ *and for all* $u \in H^1(\Omega)$ *with* $u \neq 0$.

Arguing by contradiction suppose we can find $t \in [0,1]$ and $u \in H^1(\Omega), u \neq 0$ such that

$$\left(\hat{h}_+^*\right)'_u(t,u) = 0.$$

This would give

$$\begin{aligned} &\langle A(u), h\rangle + \int_\Omega \xi(x)uhdx + \int_{\partial\Omega}\beta(x)uhd\sigma - \mu\int_\Omega u^- hdx \\ &= \lambda\int_\Omega u^+ hdx + t\int_\Omega hdx \quad \text{for all } h \in H^1(\Omega). \end{aligned} \tag{3.56}$$

If we choose $h = -u^- \in H^1(\Omega)$ in (3.56) we derive

$$\gamma(u^-) + \mu\|u^-\|_2^2 \leq 0,$$

which, due to (2.2), yields

$$c_0\|u^-\|^2 \leq 0.$$

That means $u \neq 0, u \geq 0$. So, (3.56) becomes

$$\langle A(u), h\rangle + \int_\Omega \xi(x)uhdx + \int_{\partial\Omega}\beta(x)uhd\sigma = \int_\Omega (\lambda u + t)hdx \quad \text{for all } h \in H^1(\Omega).$$

Finally, this implies that u solves the problem

$$\begin{aligned} -\Delta u + \xi(x)u &= \lambda u + t && \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(x)u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.57}$$

From (3.57) and the regularity theory of Wang [18], similar to the proof of Proposition 3.4, we have

$$u \in C^1(\overline{\Omega})_+ \setminus \{0\}.$$

Moreover, hypothesis H(ξ) and the strong maximum principle imply

$$u \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right).$$

Let $w \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ and consider the function

$$R(w,u)(x) = |\nabla w(x)|^2 - \left(\nabla u(x), \nabla\left(\frac{w^2}{u}\right)(x)\right)_{\mathbb{R}^N}.$$

Applying Picone's identity (see, for instance, Motreanu-Motreanu-Papageorgiou [12]), Green's identity and (3.57) we conclude

$$\begin{aligned}
0 &\le \int_\Omega R(w,u)dx\\
&= \|\nabla w\|_2^2 - \int_\Omega (-\Delta u)\frac{w^2}{u}dx + \int_{\partial\Omega}\beta(x)w^2 d\sigma\\
&= \|\nabla w\|_2^2 - \int_\Omega (\lambda - \xi(x))w^2dx + \int_{\partial\Omega}\beta(x)w^2d\sigma - t\int_\Omega \frac{w^2}{u}dx\\
&\le \gamma(w) - \lambda\|w\|_2^2
\end{aligned}$$

since $t \in [0,1], w, u \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. Now we choose $w = \hat{u}_1 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. This yields

$$0 \le \gamma(\hat{u}_1) - \lambda\|\hat{u}_1\|_2^2 = \left(\hat{\lambda}_1 - \lambda\right) < 0,$$

because of $\|\hat{u}_1\|_2 = 1$ and $\lambda \ge \hat{\lambda}_2 > \hat{\lambda}_1$, a contradiction. This proves Claim 2.

The homotopy invariance of the singular homology groups implies that for $r > 0$ small enough, we obtain

$$\begin{aligned}
&H_m\left(\hat{h}_+^*(0,\cdot)^0 \cap B_r, \hat{h}_+^*(1,\cdot)^0 \cap B_r\right)\\
&= H_m\left(\hat{h}_+^*(1,\cdot)^0 \cap B_r, \hat{h}_+^*(1,\cdot)^0 \cap B_r \setminus \{0\}\right) \quad \text{for all } m \in \mathbb{N}_0,
\end{aligned} \tag{3.58}$$

where $B_r = \{u \in H^1(\Omega) : \|u\| < r\}$. Claim 2 gives us

$$H_m\left(\hat{h}_+^*(1,\cdot)^0 \cap B_r, \hat{h}_+^*(1,\cdot)^0 \cap B_r \setminus \{0\}\right) = 0 \quad \text{for all } m \in \mathbb{N}_0,$$

see, for example, Motreanu-Motreanu-Papageorgiou [12, p. 160]. This in combination with (3.58) yields

$$C_m(\hat{\varphi}_+, 0) = C_m\left(\hat{\psi}_+, 0\right) = 0 \quad \text{for all } m \in \mathbb{N}_0. \tag{3.59}$$

Since $\lambda \in \left(\hat{\lambda}_k, \hat{\lambda}_{k+1}\right)$, we have $K_{\hat{\psi}_+} = \{0\}$ and so

$$C_m\left(\hat{\psi}_+, 0\right) = C_m\left(\hat{\psi}_+, \infty\right) \quad \text{for all } m \in \mathbb{N}_0,$$

see Motreanu-Motreanu-Papageorgiou [12]. Then (3.59) gives

$$C_m\left(\hat{\psi}_+, \infty\right) = 0 \quad \text{for all } m \in \mathbb{N}_0$$

and (3.55) then implies

$$C_m(\hat{\varphi}_+, \infty) = 0 \quad \text{for all } m \in \mathbb{N}_0.$$

In a similar way we show that

$$C_m(\hat{\varphi}_-, \infty) = 0 \quad \text{for all } m \in \mathbb{N}_0.$$

□

Now we can have an exact computation of the critical groups of the energy functional φ at the two constant sign solutions $u_0 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ as well as $v_0 \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ produced in Proposition 3.5. Note the fact that φ is not C^2 (recall that f is only a Carathédory function) does not permit the usage of classical results from Morse theory (see, for example, Motreanu-Motreanu-Papageorgiou [12]) and makes the computation of the critical groups of φ at $u_0 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ and at $v_0 \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ a nontrivial, interesting task.

Proposition 3.8. *Let the hypotheses H(ξ), H(β) and H(f) be satisfied and let the solutions*

$$u_0 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right), \quad v_0 \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$$

produced in Proposition 3.5 be the only nontrivial constant sign solutions of problem (1.1). Then

$$C_m(\varphi, u_0) = C_m(\varphi, v_0) = \delta_{m,1}\mathbb{Z} \quad \textit{for all } m \in \mathbb{N}_0.$$

Proof. We do the proof only for the positive solution $u_0 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$, the proof for the negative solution $v_0 \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ is very similar.

Proposition 3.4 states that

$$K_{\hat{\varphi}_+} \subseteq \operatorname{int}\left(C^1(\overline{\Omega})_+\right) \cup \{0\}$$

and by the definition of the truncation in (3.1) we know that the elements of $K_{\hat{\varphi}_+}$ are nonnegative solutions of (1.1). So, the hypothesis of the proposition implies that

$$K_{\hat{\varphi}_+} = \{0, u_0\}.$$

From the proof of Proposition 3.5 (see (3.34), (3.37)) we know that

$$\hat{\varphi}_+(0) = 0 < \hat{m}_+ \le \hat{\varphi}_+(u_0). \tag{3.60}$$

Let $\hat{\mu} < 0 < \hat{\nu} < \hat{m}_+$ and consider the following triple of sets

$$\hat{\varphi}_+^{\hat{\mu}} \subseteq \hat{\varphi}_+^{\hat{\nu}} \subseteq H^1(\Omega).$$

For this triple of sets we consider the corresponding long exact sequence of singular homology groups (see Motreanu-Motreanu-Papageorgiou [12, pp. 143, 144])

$$\begin{aligned} \cdots \longrightarrow H_m\left(H^1(\Omega), \hat{\varphi}_+^{\hat{\mu}}\right) \xrightarrow{i_*} H_m\left(H^1(\Omega), \hat{\varphi}_+^{\hat{\nu}}\right) \xrightarrow{\hat{\partial}_*} \\ \xrightarrow{\hat{\partial}_*} H_{m-1}\left(\hat{\varphi}_+^{\hat{\nu}}, \hat{\varphi}_+^{\hat{\mu}}\right) \longrightarrow \dots, \end{aligned} \tag{3.61}$$

where i_* is the group homomorphism induced by the inclusion

$$\left(H^1(\Omega), \hat{\varphi}_+^{\hat{\mu}}\right) \xhookrightarrow{i} \left(H^1(\Omega), \hat{\varphi}_+^{\hat{\nu}}\right)$$

and $\hat{\partial}_*$ is the composed boundary homomorphism.

The rank theorem implies

$$\begin{aligned} \operatorname{rank} H_m\left(H^1(\Omega), \hat{\varphi}_+^{\hat{\nu}}\right) &= \operatorname{rank}\ker \hat{\partial}_* + \operatorname{rank}\operatorname{im} \hat{\partial}_* \\ &= \operatorname{rank}\operatorname{im} i_* + \operatorname{rank}\operatorname{im} \hat{\partial}_* \end{aligned} \tag{3.62}$$

because of the exactness in (3.61). Recall that $\hat{\nu} \in (0, \hat{m}_+)$ and $K_{\hat{\varphi}_+} = \{0, u_0\}$. Therefore, from (3.60) and Proposition 6.61 in Motreanu-Motreanu-Papageorgiou [12] we obtain

$$H_m\left(H^1(\Omega), \hat{\varphi}_+^{\hat{\nu}}\right) = C_m\left(\hat{\varphi}_+, u_0\right) \quad \text{for all } m \in \mathbb{N}_0. \tag{3.63}$$

Since $\hat{\mu} < 0$ and $K_{\hat{\varphi}_+} = \{0, u_0\}$, from (3.50) it follows that

$$H_m\left(H^1(\Omega), \hat{\varphi}_+^{\hat{\nu}}\right) = C_m\left(\hat{\varphi}_+, \infty\right) = 0 \quad \text{for all } m \in \mathbb{N}_0, \tag{3.64}$$

cf. Proposition 3.7. Finally, Lemma 6.55 of Motreanu-Motreanu-Papageorgiou [12] gives

$$H_{m-1}\left(\hat{\varphi}_+^{\hat{\nu}}, \hat{\varphi}_+^{\hat{\mu}}\right) = C_{m-1}(\hat{\varphi}_+, 0) = \delta_{m-1,0}\mathbb{Z} = \delta_{m,1}\mathbb{Z} \quad \text{for all } m \in \mathbb{N}_0, \tag{3.65}$$

see also Proposition 3.3.

We return back to (3.62) and use (3.63), (3.64) as well as (3.65). This leads to

$$\operatorname{rank} C_1\left(\hat{\varphi}_+, u_0\right) \leq 1. \tag{3.66}$$

From the proof of Proposition 3.5 we know that $u_0 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ is a critical point of mountain pass type of the functional $\hat{\varphi}_+$. Hence

$$C_1\left(\hat{\varphi}_+, u_0\right) \neq 0,$$

(see Motreanu-Motreanu-Papageorgiou [12, p. 168]) which means

$$\operatorname{rank} C_1\left(\hat{\varphi}_+, u_0\right) \geq 1. \tag{3.67}$$

Note that in (3.61) only the tail (that is, $m = 1$) is nontrivial (see (3.64), (3.65)). Finally, from (3.66) and (3.67) we infer that

$$C_m\left(\hat{\varphi}_+, u_0\right) = \delta_{m,1}\mathbb{Z} \quad \text{for all } m \in \mathbb{N}_0. \tag{3.68}$$

Now, let us consider the homotopy $\tilde{h}_+ : [0,1] \times H^1(\Omega) \to \mathbb{R}$ defined by

$$\tilde{h}_+(t, u) = (1-t)\varphi(u) + t\hat{\varphi}_+(u).$$

Suppose that we can find $(t_n)_{n\geq 1} \subseteq [0,1]$ and $(u_n)_{n \geq 1} \subseteq H^1(\Omega)$ such that

$$t_n \to t, \quad u_n \to u_0, \quad \left(\tilde{h}_+\right)'_u (t_n, u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.69}$$

The last assertion in (3.69) implies

$$\begin{aligned} -\Delta u_n + \xi(x) u_n &= f(x, u_n^+) + (1-t_n) f(x, -u_n^-) + t_n \mu u_n^- && \text{in } \Omega, \\ \frac{\partial u_n}{\partial n} + \beta(x) u_n &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.70}$$

From (3.70) in combination with the growth condition on f stated in (3.31) and the regularity theory of Wang [18] we know that we can find $\alpha \in (0,1)$ and $c_{11} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}), \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_{11} \quad \text{for all } n \in \mathbb{N}. \tag{3.71}$$

From (3.71) and the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we infer that

$$u_n \to u_0 \text{ in } C^1(\overline{\Omega}),$$

see (3.69). Recall that $u_0 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ and $\operatorname{int}\left(C^1(\overline{\Omega})_+\right) \subseteq C^1(\overline{\Omega})$ is open, we obtain

$$u_n \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right) \quad \text{for all } n \geq n_0.$$

Therefore, $(u_n)_{n\geq 1}$ is a sequence of distinct positive solutions of (1.1), a contradiction. Hence, (3.69) cannot occur and so from the homotopy invariance of critical groups (see, for example, Gasiński-Papageorgiou [9, p. 836]) we have

$$C_m\left(\tilde{h}_+(0,\cdot), u_0\right) = C_m\left(\tilde{h}_+(1,\cdot), u_0\right) \quad \text{for all } m \in \mathbb{N}_0.$$

This gives because of (3.68)

$$C_m\left(\varphi, u_0\right) = C_m\left(\hat{\varphi}_+, u_0\right) = \delta_{m,1}\mathbb{Z} \quad \text{for all } m \in \mathbb{N}_0.$$

Similarly we can show that

$$C_m\left(\varphi, v_0\right) = \delta_{m,1}\mathbb{Z} \quad \text{for all } m \in \mathbb{N}_0.$$

□

Now we are ready to produce a third nontrivial smooth solution.

Proposition 3.9. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ are satisfied, then problem* (1.1) *has a third nontrivial smooth solution*

$$y_0 \in C^1(\overline{\Omega}), \quad y_0 \notin \{0, u_0, v_0\}.$$

Proof. From Proposition 3.6 we have

$$C_m(\varphi, \infty) = \delta_{m,d_k}\mathbb{Z} \quad \text{for all } m \in \mathbb{N}_0 \text{ with } d_k = \dim \bigoplus_{i=1}^{k} E\left(\hat{\lambda}_i\right).$$

Then from Motreanu-Motreanu-Papageorgiou [12, Theorem 6.62, p. 160] (an easy consequence of the Morse relation given in (2.6)) we know that there exists $y_0 \in K_\varphi$ such that

$$C_{d_k}(\varphi, y_0) \neq 0. \tag{3.72}$$

From Propositions 3.3 and 3.8 we have

$$C_m(\varphi, 0) = \delta_{m,0}\mathbb{Z} \quad \text{and} \quad C_m(\varphi, u_0) = C_m(\varphi, v_0) = \delta_{m,1}\mathbb{Z} \tag{3.73}$$

for all $m \in \mathbb{N}_0$. Since $d_k \geq 2$ (recall that $k \geq 2$), from (3.72) and (3.73) it follows that

$$y_0 \notin \{0, u_0, v_0\}.$$

Because of $y_0 \in K_\varphi$, we know that y_0 is a third nontrivial solution of (1.1) and the regularity theory of Wang [18] implies that $y_0 \in C^1(\overline{\Omega})$. □

Theorem 3.10. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ are satisfied, then problem* (1.1) *admits at least three nontrivial smooth solutions*

$$u_0 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right), \quad v_0 \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right) \quad \textit{and} \quad y_0 \in C^1(\overline{\Omega}).$$

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E-mail address: npapg@math.ntua.gr
E-mail address: winkert@math.tu-berlin.de