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Research paper

Normalized solutions to Schrödinger systems with critical nonlinearities

Yuxi Meng [a], Xiaoming He [b], Patrick Winkert [c,*]

[a] School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China
[b] College of Science, Minzu University of China, Beijing, 100081, China
[c] Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, 10623 Berlin, Germany

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ABSTRACT

We consider a system of coupled Schrödinger equations involving critical exponent given by

$$\begin{cases} -\Delta u + \lambda_1 u = \mu |u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu |v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \mathbb{R}^N. \end{cases}$$

We study the existence of positive ground state solutions having prescribed mass

$$\int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x = a_1^2 \quad \text{and} \quad \int_{\mathbb{R}^N} |v|^2\,\mathrm{d}x = a_2^2,$$

where $N = 3,4$, $a_1, a_2 > 0$, $q \in (2, 2^*)$, $\alpha, \beta > 1$ with $\alpha + \beta = 2^* = \frac{2N}{N-2}$, the Sobolev critical exponent, $\lambda_1, \lambda_2 \in \mathbb{R}$ are parameters to be specified and will appear as Lagrange multipliers, and $\mu > 0$ is a parameter. Under some L^2-subcritical, L^2-critical and L^2-supercritical perturbations $\mu|u|^{q-2}u$ and $\mu|v|^{q-2}v$, respectively, we prove several existence results by using variational methods, which can be considered as a counterpart of the Brézis-Nirenberg problem in the context of normalized solutions for coupled Schrödinger equations. Our results extend and improve the existing literature in several directions.

1. Introduction and main results

This paper is concerned with the existence of solutions $(\lambda_1, \lambda_2, u, v) \in \mathbb{R}^2 \times H^1(\mathbb{R}^N, \mathbb{R}^2)$ to the following Schrödinger system with critical growth

$$\begin{cases} -\Delta u + \lambda_1 u = \mu |u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu |v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

with the prescribed L^2-norm

$$\int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x = a_1^2 \quad \text{and} \quad \int_{\mathbb{R}^N} |v|^2\,\mathrm{d}x = a_2^2, \tag{1.2}$$

* Corresponding author.
E-mail addresses: yxmeng125@163.com (Y. Meng), xmhe923@muc.edu.cn (X. He), winkert@math.tu-berlin.de (P. Winkert).

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where $N = 3, 4$, $\mu, a_1, a_2 > 0$ are positive constants, $\alpha, \beta > 1$ satisfy $\alpha + \beta = 2^*$ with $2^* = \frac{2N}{N-2}$ being the Sobolev critical exponent. We refer to this type of solutions as to normalized solutions, since (1.2) imposes a normalization on the L^2-masses of u and v. Under this circumstance, λ_1 and λ_2 cannot be determined a priori, but are part of the unknowns. The problem to be investigated comes from the research of solitary waves for the system of coupled Schrödinger equations

$$\begin{cases} -i\dfrac{\partial \Psi_1}{\partial t} = \Delta \Psi_1 + \mu|\Psi_1|^{q-2}\Psi_1 + \dfrac{2\alpha}{\alpha+\beta}|\Psi_1|^{\alpha-2}\Psi_1|\Psi_2|^{\beta} & \text{in } \mathbb{R}^N, \\ -i\dfrac{\partial \Psi_2}{\partial t} = \Delta \Psi_2 + \mu|\Psi_2|^{q-2}\Psi_2 + \dfrac{2\beta}{\alpha+\beta}|\Psi_1|^{\alpha}|\Psi_2|^{\beta-2}\Psi_2 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $\Psi_j = \Psi_j(x,t) \in \mathbb{C}$, $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $j = 1, 2$. System (1.3) can be used to model various physical phenomena, such as binary mixtures of Bose–Einstein condensates or the propagation of mutually incoherent wave packets in nonlinear optics, see, for example, Esry–Greene–Burke–Bohn [1], Frantzeskakis [2] and Timmermans [3] for more applied backgrounds. An important and of course well-known feature of (1.3) is the conservation of three quantities: the energy

$$\begin{aligned} J_{\mathbb{C}}(\Psi_1,\Psi_2) = {} & \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\Psi_1|^2 + |\nabla\Psi_2|^2 \,\mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^N} |\Psi_1|^q + |\Psi_2|^q \,\mathrm{d}x \\ & - \frac{2}{\alpha+\beta}\int_{\mathbb{R}^N} |\Psi_1|^{\alpha}|\Psi_2|^{\beta}\,\mathrm{d}x, \end{aligned}$$

and the masses

$$\int_{\mathbb{R}^N} |\Psi_1|^2 \,\mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}^N} |\Psi_2|^2 \,\mathrm{d}x.$$

The L^2-norms $|\Psi_1(\cdot,t)|_2, |\Psi_2(\cdot,t)|_2$ of solutions are independent of $t \in \mathbb{R}$ and have a clear physical meaning. In the contexts mentioned above, they represent the number of particles of each component in Bose–Einstein condensates or the energy power supply in the context of nonlinear optics.

A solitary wave of (1.3) is a solution having the form

$$\Psi_1(x,t) = e^{i\lambda_1 t}u(x) \quad \text{and} \quad \Psi_2(x,t) = e^{i\lambda_2 t}v(x),$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u, v \in H^1(\mathbb{R}^N)$ are time independent real-valued functions solving (1.1). There are two different approaches to finding for solutions to (1.1): On the one hand, one can consider the frequencies of $\lambda_1 \in \mathbb{R}$ and $\lambda_2 \in \mathbb{R}$ as fixed, on the other hand, one can include them in the unknown and prescribe the masses. In the latter case, $\lambda_1, \lambda_2 \in \mathbb{R}$ are unknown quantities that appear as Lagrange multipliers with respect to the mass constraint.

We note that if $\alpha = \beta = 2$, $q = 4$ and $N = 3$, the system (1.1) is related to the following coupled system of Schrödinger equations

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2 & \text{in } \mathbb{R}^3, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^3. \end{cases} \tag{1.4}$$

The problem (1.4) with fixed λ_1, λ_2 has been studied by many authors in the last two decades. In this case, we refer the interested reader to the papers of Ambrosetti-Colorado [4], Bartsch-Dancer-Wang [5], Bartsch-Wang [6], Bartsch-Wang-Wei [7], Chen-Zou [8,9], Gou-Jeanjean [10], Lin-Wei [11], Sirakov [12] and related references therein. In contrast, there are not many papers that investigate the existence of normalized solutions for (1.2)–(1.4). In [13], Bartsch-Jeanjean-Soave proved the existence of positive normalized solutions for different ranges of the coupling parameter $\beta > 0$, without any assumption on the masses a_1, a_2. Bartsch-Soave [14] showed the existence of normalized solutions to (1.2)–(1.4) for any given $\mu_1, \mu_2, a_1, a_2 > 0$ and $\beta < 0$. They also investigated the phenomenon of phase separation for the solutions as $\beta \to -\infty$. In [15], Bartsch-Soave proved the existence of infinitely many normalized solutions of (1.2)–(1.4) with $a_1 = a_2 = a$ and $\mu_1 = \mu_2 = \mu$ by using the Krasnoselskii genus approach for the constrained functional. In [16], Bartsch-Zhong-Zou studied the existence and non-existence of normalized solutions to (1.2)–(1.4) by an approach based on the fixed point index in cones, bifurcation theory, and the continuation method. Gou-Jeanjean [17] considered the existence of multiple positive solutions of (1.1)–(1.2) with $2 < \alpha + \beta < 2^*$, one solution is a local minimizer, the other one is obtained through a constrained mountain-pass and a constrained linking, respectively. Recently, Bartsch-Li-Zou [18] proved the existence and asymptotic properties of normalized ground states of (1.1)–(1.2) with critical exponent $q = 2^*$ with $N \in \{3, 4\}, 2 < \alpha + \beta < 2^*$. Jeanjean-Zhang-Zhong [19] obtained the normalized ground states for (1.1)–(1.2) with mass super-critical growth $2 + \frac{4}{N} < q, \alpha + \beta < 2^*$, and $N \in \{1, 2, 3, 4\}$. Recently, Mederski-Schino [20] studied the existence of least energy solutions to a cooperative systems of coupled Schrödinger equations with general nonlinearities of the form

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \partial_i G(u) & \text{in } \mathbb{R}^N, \\ u_i \in H^1(\mathbb{R}^N), \quad i \in \{1, 2, \dots, K\}, \\ \int_{\mathbb{R}^N} |u_i|^2 \,\mathrm{d}x \le \rho_i^2, \end{cases} \tag{1.5}$$

with $G \ge 0$, where $\rho_i > 0$ is prescribed and $(\lambda_i, u_i) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ is to be determined, $i \in \{1, \dots, K\}$. The authors established several existence results of normalized solutions for problem (1.5). The main innovation of [20] is based on the minimization of the energy functional over a linear combination of the Nehari and Pohozaev constraints intersected with the product of closed balls in $L^2(\mathbb{R}^N)$ of radii ρ_i, which allows to provide general growth assumptions about G and to know in advance the sign of the corresponding

Lagrange multipliers. For existence results of normalized solutions to Schrödinger equations or systems in bounded domains, we refer to the works of Noris-Tavares-Verzini [21,22] and Pierotti-Verzini [23], see also the references therein.

If $u = v$ and $\lambda_1 = \lambda_2$, the system (1.1)–(1.2) reduces to the following problem

$$\begin{cases} -\Delta u = \lambda u + g(u), & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x = a^2, & u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.6}$$

where g is the nonlinearity. In the outstanding paper [24], Jeanjean studied the existence of solutions of (1.6) with nonlinearity of the type $g(s) = \sum_{i=1}^m a_i |s|^{\sigma_i} s$ with $a_i > 0, 0 < \sigma_i < \frac{N}{N-2}$ for $N \geq 3$ and $\sigma_i > 0$ if $N = 1, 2$ and $i = 1, \dots, m$ with $m \in \mathbb{N}$. The occurrence of the L^2-constraint renders several methods developed to deal with variational problems without constraints useless, and the L^2-constraint induces a new critical exponent, the L^2-critical exponent given by

$$\overline{q} := 2 + \frac{4}{N},$$

and the number $\overline{q}$ can keep the mass invariant by the law of conservation of mass. Precisely for this reason, $2 + \frac{4}{N}$ is called L^2-critical exponent or mass critical exponent, which is the threshold exponent for many dynamical properties such as global existence, blow-up, stability or instability of ground states. In particular, it strongly influences the geometrical structure of the corresponding functional. In 2020, Soave [25,26] started the research of the existence and properties of ground states for problem (1.6) with combined power type nonlinearities $g(u) = \mu|u|^{q-2}u + |u|^{p-2}u$ with $2 < q < p \leq 2^*$. He presented a complete classification on the existence and nonexistence of normalized solutions that let q be L^2-subcritical, L^2-critical and L^2-supercritical. Since then, the existence and properties of these normalized solutions for Schrödinger equations or systems have attracted the attention of more and more researchers in recent years. For further studies on this aspect, we refer to Bartsch-de Valeriola [27], Bartsch-Molle-Rizzi-Verzini [28], Hirata-Tanaka [29], Jeanjean-Le [30], Jeanjean-Lu [31], Wei-Wu [32] and the references therein.

The purpose of this paper is to study the existence of normalized solutions of problem (1.1)–(1.2). We present several existence results in the following three cases:

(i) L^2-subcritical case: $2 < q < \overline{q}$;
(ii) L^2-supercritical case: $\overline{q} < q < 2^*$;
(iii) L^2-critical case: $q = \overline{q}$

This study is a new contribution regarding existence of normalized ground states for the Sobolev critical nonlinear Schrödinger system in the whole space $\mathbb{R}^N$, which improves and complements the studies of Alves-de Morais Filho-Souto [33], Han [34] and Hsu-Lin [35], which are concerned with the existence of solutions of (1.1) in bounded domains $\Omega \subset \mathbb{R}^N$ without prescribed L^2-norm, while in this paper we are concerned with the existence of normalized solutions. By studying the geometrical structure of the corresponding Pohozaev manifold, we have obtained a constrained Palais–Smale sequence with additional properties, and show the compactness of this special constrained Palais–Smale sequence at some energy level. As far as we know, normalized solutions to (1.1)–(1.2) with critical exponent have not yet been considered in the literature.

In order to search solutions to (1.1)–(1.2), we introduce the corresponding energy functional given by

$$I_\mu(u, v) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right)\,\mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^N} \left(|u|^q + |v|^q\right)\,\mathrm{d}x - \frac{2}{2^*}\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta\,\mathrm{d}x,$$

under the constraint $S_{a_1} \times S_{a_2}$, where

$$S_a = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x = a^2 \right\}.$$

It is standard to check that I_μ is of class C^1 in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, and any critical point (u, v) of $I_\mu|_{S_{a_1}\times S_{a_2}}$ corresponds to a solution to (1.1) satisfying (1.2). Here the parameters $\lambda_1, \lambda_2 \in \mathbb{R}$ arise as Lagrangian multipliers. In particular, we are interested in ground state solutions which are defined in the following way:

Definition 1.1. We say that (u_0, v_0) is a normalized ground state of system (1.1)–(1.2), if it is a solution to (1.1)–(1.2) having minimal energy among all the normalized solutions. Namely,

$$I_\mu(u_0, v_0) = \inf\{I_\mu(u, v) : (u, v) \text{ solves (1.1)–(1.2) for some } (\lambda_1, \lambda_2) \in \mathbb{R}^2\}.$$

This definition seems particularly suitable in our context, since I_μ is unbounded from below on $S_{a_1} \times S_{a_2}$ and therefore no global minima exist.

Now, we formulate the main results of this paper. First, we have the following result in the L^2-subcritical case $2 < q < \overline{q} := 2 + \frac{4}{N}$.

Theorem 1.1. *Let $N = 3, 4$, $a_1, a_2 > 0$, $\alpha, \beta > 1$ such that $\alpha + \beta = 2^*$ and $q \in (2, 2 + \frac{4}{N})$. Then, $I_\mu|_{S_{a_1}\times S_{a_2}}$ has a ground state (u, v) which is a positive, radially symmetric function and solves problem (1.1)–(1.2) for some $\lambda_1, \lambda_2 > 0$, provided $0 < \mu < \min\{\mu_1, \mu_2\}$, where μ_1, μ_2 are explicitly defined in (3.1)–(3.2) below. Furthermore, the normalized ground state is a local minimizer of $I_\mu(u, v)$ on $S_{a_1} \times S_{a_2}$.*

The next two theorems are concerned with the L^2-supercritical/critical cases $2 + \frac{4}{N} \leq q < 2^*$ by constructing the mountain-pass type ground states, if the parameter $\mu > 0$ sufficiently small.

Theorem 1.2. *Let $N = 3, 4$, $a_1, a_2 > 0$, $\alpha, \beta > 1$ such that $\alpha + \beta = 2^*$ and $q \in (2 + \frac{4}{N}, 2^*)$. Then $I_\mu|_{S_{a_1} \times S_{a_2}}$ has a ground state (u, v) which is a positive, radially symmetric function and solves problem (1.1)–(1.2) for some $\lambda_1, \lambda_2 > 0$. Moreover, $0 < m_\mu(a_1, a_2) < \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}$, where $m_\mu(a_1, a_2)$ is given in (2.8), and (u, v) is a mountain-pass type solution.*

Theorem 1.3. *Let $N = 3, 4$, $a_1, a_2 > 0$, $\alpha, \beta > 1$ such that $\alpha + \beta = 2^*$ and $q = 2 + \frac{4}{N}$. Then $I_\mu|_{S_{a_1} \times S_{a_2}}$ has a ground state (u, v) which is a positive, radially symmetric function and solves problem (1.1)–(1.2) for some $\lambda_1, \lambda_2 > 0$, provided $0 < \mu < \mu_3$, where μ_3 is explicitly defined in (5.1) below. Moreover, $0 < m_\mu(a_1, a_2) < \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}$ and (u, v) is a mountain-pass type solution.*

Finally, we present another existence result for (1.1)–(1.2) in the L^2-supercritical case $2 + \frac{4}{N} < q < 2^*$ when $\mu > 0$ is sufficiently large.

Theorem 1.4. *Let $N = 3, 4$, $a_1, a_2 > 0$, $\alpha, \beta > 1$ such that $\alpha + \beta = 2^*$ and $q \in (2 + \frac{4}{N}, 2^*)$. Then there exists $\mu^* = \mu^*(a_1, a_2) > 0$ such that for any $\mu \geq \mu^*$, problem (1.1)–(1.2) possesses a positive, radially symmetric solution (u, v) for some $\lambda_1, \lambda_2 > 0$.*

Remark 1.5. Note that in the paper by Mederski-Schino [20] the nonlinearity G can be assumed to have at least L^2-critical growth at 0 and to be Sobolev critical growth. In particular, in the Sobolev critical growth, the nonlinearity G takes the form

$$G(u) = \widetilde{G}(u) + \frac{1}{2^*}\sum_{j=1}^{K} \theta_j |u_j|^{2^*},$$

where $\widetilde{G}(\cdot)$ is a subcritical perturbation. The energy of the minimizer in the constrained set of (1.5) is strictly less than the number $\frac{1}{N} S^{N/2} \sum_{j=1}^{K} \theta_j^{1-N/2}$. But in our paper, the nonlinearity appears differently from (1.5) as the entangled mixed item

$$G(u) = \frac{2}{2^*}|u|^\alpha |v|^\beta, \quad \alpha + \beta = 2^*.$$

Moreover, the energy functional $I_\mu(u, v)$ of (1.1) and (1.2) satisfies the Palais–Smale compactness condition below the minimum threshold value $\frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}$. So, our paper and the one by Mederski-Schino [20] have their own characteristics and interests in theoretical methods, independently of each other.

When searching for normalized solutions for (1.1)–(1.2), one of the main difficulties is the lack of compactness of the constrained Palais–Smale sequences. Indeed, since the embeddings $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ and $H^1_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ are not compact, it is hard to verify whether the weak limits of the constrained Palais–Smale sequence lie in the constraint $S_{a_1} \times S_{a_2}$. To overcome this difficulty, we adopt the idea of Jeanjean [24] by showing that the mountain-pass geometry of $I_\mu|_{S_{a_1} \times S_{a_2}}$ allows to construct a Palais–Smale sequence of functions satisfying the Pohozaev identity, which yields boundedness and is useful to prove the strong H^1-convergence. Another difficulty, as naturally expected, is the presence of the critical Sobolev term in (1.1), which further complicates the study of convergence of constrained Palais–Smale sequences. To overcome such a difficulty, we will perform a careful analysis of the behavior of the constrained Palais–Smale sequences to analyze the possible reason of lack of compactness and to find out the regions of the energy levels where the Palais–Smale condition is satisfied and compactness can be restored. For this purpose, the concentration-compactness principle (see Alves-de Morais Filho-Souto [33] and Han [34]) and the mountain-pass theorem (see, for example, Willem [36]) are involved to obtain both ground state solutions to (1.1)–(1.2) by minimizing the functional on the associated Pohozaev manifold and mountain-pass solutions.

The paper is organized as follows. In Section 2, we start with some preliminary results which will be frequently used in the rest of the paper while Section 3 presents the proof of Theorem 1.1, which is about the L^2-subcritical case. Section 4 is devoted to the proof of Theorem 1.2 with the L^2-supercritical perturbation and in Section 5, we prove Theorem 1.3 for the L^2-critical case. Finally, by using the concentration-compactness principle, we give the proof of Theorem 1.4 in Section 6.

2. Preliminaries

In this section we recall the main notations and tools that will be needed in the sequel. For $1 < s < \infty$ we denote by $L^s(\mathbb{R}^N)$ the usual Lebesgue spaces with norm

$$\|u\|_s := \left(\int_{\mathbb{R}^N} |u|^s \,\mathrm{d}x\right)^{\frac{1}{s}}$$

and $H^1 = H^1(\mathbb{R}^N)$ is the standard Hilbert space equipped with norm and inner product given by

$$\|u\|^2_{H^1} = \langle u, u\rangle \quad \text{and} \quad \langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)\,\mathrm{d}x.$$

Further, let $H = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and H^1_r denotes the subspace of functions in H^1 which are radial symmetric with respect to 0, and $H_r = H^1_r \times H^1_r$ as well as $S_{a,r} = S_a \cap H^1_r$. By $B_R(y)$ we denote the ball centered at y with radius R, $B_R := B_R(0)$ and $\|\cdot\|$ denotes the norm in H^1 or H. Moreover, u^* is a rearrangement of $|u|$. We recall that

$$\|\nabla u^*\|_2 \leq \|\nabla u\|_2, \quad \|u^*\|_s = \|u\|_s \quad \text{and} \quad \int_{\mathbb{R}^N} |u^*|^p |v^*|^q \,\mathrm{d}x \geq \int_{\mathbb{R}^N} |u|^p |v|^q \,\mathrm{d}x,$$

see, for example, Lieb-Loss [37].

Positive constants whose exact values are not important in the relevant arguments, and that may vary from line to line, are generally denoted by C or C_i, where $i \in \mathbb{N}$. For $N \geq 3, q \in (2, 2^*]$, we recall the following Gagliardo–Nirenberg inequality, see Nirenberg [38]:

$$\|u\|_q^q \leq C_{N,q} \|\nabla u\|_2^{q\gamma_q} \|u\|_2^{q(1-\gamma_q)} \quad \text{for all } u \in H^1(\mathbb{R}^N), \tag{2.1}$$

where the optimal constant $C_{N,q}$ depends on N, q, and the number

$$\gamma_q := \frac{N(q-2)}{2q} \quad \text{for all } q \in (2, 2^*]. \tag{2.2}$$

Particularly, if $q = 2^*$, then we denote by S be the best Sobolev constant defined by

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}, \tag{2.3}$$

where the Sobolev space $D^{1,2}(\mathbb{R}^N)$ is defined as the completion of the space $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{D^{1,2}(\mathbb{R}^N)} := \|\nabla u\|_2$. Using (2.2), it is easy to see that

$$q\gamma_q \begin{cases} < 2, & \text{if } 2 < q < \overline{q}, \\ = 2, & \text{if } q = \overline{q} \text{ and } \gamma_{2^*} = 1, \\ > 2, & \text{if } \overline{q} < q < 2^*. \end{cases}$$

When studying Schrödinger systems, we need a vector-valued version of the Gagliardo–Nirenberg inequality. For $\alpha, \beta > 1$, $2 < \alpha + \beta \leq 2^*$, we define

$$Q(u, v) := \frac{\left(\|u\|_2^2 + \|v\|_2^2\right)^{(\alpha+\beta-(\alpha+\beta)\gamma_{\alpha+\beta})/2} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{(\alpha+\beta)\gamma_{\alpha+\beta}/2}}{\left\| |u|^\alpha |v|^\beta \right\|_1}$$

and by Correia [39], we have

$$C(N, \alpha, \beta)^{-1} := \inf_{u,v \in H^1(\mathbb{R}^N)\setminus\{0\}} Q(u, v) > 0. \tag{2.4}$$

From (2.4), we get the vector-valued Gagliardo–Nirenberg inequality in the form

$$\begin{aligned} &\| |u|^\alpha |v|^\beta \|_1 \\ &\leq C(N, \alpha, \beta) \left(\|u\|_2^2 + \|v\|_2^2\right)^{(\alpha+\beta-(\alpha+\beta)\gamma_{\alpha+\beta})/2} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{(\alpha+\beta)\gamma_{\alpha+\beta}/2}, \end{aligned}$$

which holds for $u, v \in H^1(\mathbb{R}^N)$. The vector-valued Gagliardo–Nirenberg inequality has been investigated by many authors. We refer to the papers of Correia [39] and Ma-Zhao [40] and the references therein. Particularly, if $\alpha + \beta = 2^*$, we define

$$S_{\alpha,\beta} = \inf_{u,v \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) \mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, \mathrm{d}x\right)^{2/2^*}}. \tag{2.5}$$

From Alves-de Morais Filho-Souto [33, Theorem 5], we have that

$$S_{\alpha,\beta} = \left(\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right) S, \tag{2.6}$$

where S is the best constant defined by (2.3).

Let

$$\mathcal{P}_{a_1,a_2} = \{(u, v) \in S_{a_1} \times S_{a_2} : P(u, v) = 0\},$$

where

$$\begin{aligned} P(u, v) = &\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) \mathrm{d}x - \mu\gamma_q \int_{\mathbb{R}^N} \left(|u|^q + |v|^q\right) \mathrm{d}x \\ &- 2\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, \mathrm{d}x. \end{aligned} \tag{2.7}$$

As proved by Bartsch-Jeanjean-Soave [13, Lemma 4.6], any solution of (1.4) belongs to $\mathcal{P}_{a_1,a_2}$. The equation $P(u, v) = 0$ being the Pohozaev identity for (1.1) and the Pohozaev manifold $\mathcal{P}_{a_1,a_2}$ will play an important role in our proofs. Therefore, if (u, v) solves system (1.1) for some λ_1, λ_2 and $(u, v) \in \mathcal{P}_{a_1,a_2}$ is a minimizer of the constraint minimization

$$m_\mu(a_1, a_2) = \inf_{(u,v) \in \mathcal{P}_{a_1,a_2}} I_\mu(u, v), \tag{2.8}$$

then (u,v) has least energy among all the solutions of (1.1) and (1.2). Namely, (u,v) is a normalized ground state solution to (1.1) and (1.2).

For $u \in S_a$ and $s \in \mathbb{R}$, we define the function

$$(s \star u)(x) := e^{\frac{Ns}{2}} u(e^s x) \quad \text{for all } x \in \mathbb{R}^N.$$

It is straightforward to check that if $u \in S_a$, then $s \star u \in S_a$ for every $s \in \mathbb{R}$. We define $s \star (u,v) = (s \star u, s \star v)$ and the fiber map as

$$\begin{aligned}\Phi_{(u,v)}(s) &:= I_\mu(s \star (u,v)) \\ &= \frac{e^{2s}}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + |\nabla v|^2\right)\mathrm{d}x - \frac{\mu e^{q\gamma_q s}}{q}\int_{\mathbb{R}^N}\left(|u|^q + |v|^q\right)\mathrm{d}x \\ &\quad - \frac{2e^{2^* s}}{2^*}\int_{\mathbb{R}^N}|u|^\alpha |v|^\beta\,\mathrm{d}x.\end{aligned}$$

An easy computation shows that $\Phi'_{(u,v)}(s) = P(s \star (u,v))$ and

$$\mathcal{P}_{a_1,a_2} = \{(u,v) \in S_{a_1} \times S_{a_2} : \Phi'_{(u,v)}(0) = 0\}.$$

In this spirit, we split the manifold $\mathcal{P}_{a_1,a_2}$ into the disjoint union

$$\mathcal{P}_{a_1,a_2} = \mathcal{P}^+_{a_1,a_2} \cup \mathcal{P}^0_{a_1,a_2} \cup \mathcal{P}^-_{a_1,a_2},$$

where

$$\begin{aligned}\mathcal{P}^+_{a_1,a_2} &:= \left\{(u,v) \in \mathcal{P}_{a_1,a_2} : \Phi''_{(u,v)}(0) > 0\right\}, \\ \mathcal{P}^0_{a_1,a_2} &:= \left\{(u,v) \in \mathcal{P}_{a_1,a_2} : \Phi''_{(u,v)}(0) = 0\right\}, \\ \mathcal{P}^-_{a_1,a_2} &:= \left\{(u,v) \in \mathcal{P}_{a_1,a_2} : \Phi''_{(u,v)}(0) < 0\right\},\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\Phi''_{(u,v)}(0) &= 2\int_{\mathbb{R}^N}|\nabla u|^2 + |\nabla v|^2\,\mathrm{d}x - \mu q\gamma_q^2\int_{\mathbb{R}^N}|u|^q + |v|^q\,\mathrm{d}x \\ &\quad - 22^*\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x.\end{aligned}$$

In order to prove our results, we need the monotonicity and convexity of $\Phi_{(u,v)}(s)$, which will strongly affect the structure of $\mathcal{P}_{a_1,a_2}$ and thus have a strong effect on the minimization problem (2.8). The following lemma can be found in Strauss [41].

Lemma 2.1. *Let $N \ge 3$. Then the embedding $H^1_r(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N)$ is compact for any $2 < t < 2^*$.*

Now, we recall the following version of the Brézis-Lieb lemma, see Chen-Zou [9, Lemma 2.3].

Lemma 2.2. *Let $N \ge 3$, $\alpha, \beta > 1$ and $2 \le \alpha + \beta \le 2^*$. If $(u_n, v_n)_{n\in\mathbb{N}} \subseteq H$ is a sequence such that $(u_n, v_n) \rightharpoonup (u,v)$ in H, then (up to a subsequence if necessary)*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|u_n|^\alpha |v_n|^\beta - |u|^\alpha|v|^\beta - |u_n - u|^\alpha |v_n - v|^\beta\right)\mathrm{d}x = 0.$$

Furthermore, we need to generalize the concentration-compactness principle to the case of systems, see Han [34] and Long-Yang [42].

Lemma 2.3. *Let $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subseteq D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ be a sequence such that $u_n \rightharpoonup u$, $v_n \rightharpoonup v \in D^{1,2}(\mathbb{R}^N)$. Assume that $|\nabla u_n|^2 + |\nabla v_n|^2 \rightharpoonup \omega$ and $|u_n|^\alpha |v_n|^\beta \rightharpoonup \nu$ weakly in the sense of measures. Then, there exist some at most countable set J, a family of points $\{x_j\}_{j\in J} \subset \mathbb{R}^N$ and families of positive numbers $\{\nu_j\}_{j\in J}$ and $\{\omega_j\}_{j\in J}$ such that*

$$\begin{aligned}\nu * &= |u|^\alpha |v|^\beta + \sum_{j\in J}\nu_j \delta_{x_j}, \\ \omega &\ge |\nabla u|^2 + |\nabla v|^2 + \sum_{j\in J}\omega_j\delta_{x_j}, \\ \omega_j &\ge S_{\alpha,\beta}\nu_j^{\frac{2}{\alpha+\beta}},\end{aligned}$$

where δ_x is the Dirac-mass of mass 1 concentrated at $x \in \mathbb{R}^N$.

Lemma 2.4. *Let $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subseteq D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ be a sequence as in Lemma 2.3 and define*

$$\omega_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|\ge R}\left(|\nabla u_n|^2 + |\nabla v_n|^2\right)\mathrm{d}x,$$

$$\nu_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|\ge R}|u_n|^\alpha|v_n|^\beta\,\mathrm{d}x.$$

Then it follows that

$$\omega_\infty \ge S_{\alpha,\beta}\nu_\infty^{\frac{2}{\alpha+\beta}},$$

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2+|\nabla v_n|^2\right)\mathrm{d}x=\int_{\mathbb{R}^N}d\omega+\omega_\infty,$$

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\,\mathrm{d}x=\int_{\mathbb{R}^N}d\nu+\nu_\infty.$$

Fix $\mu>0$, the following compactness lemma will play a crucial role in the sequel.

Lemma 2.5. *Assume that*

$$m_\mu(a_1,a_2)\le m_\mu\left(b_1,b_2\right)\quad \textit{for any } 0<b_1\le a_1, 0<b_2\le a_2. \tag{2.10}$$

Let $\{(u_n,v_n)\}_{n\in\mathbb{N}}\subseteq S_{a_1}\times S_{a_2}$ *be a sequence consisting of radially symmetric functions such that, as* $n\to+\infty$,

$$I'_\mu\left(u_n,v_n\right)+\lambda_{1,n}u_n+\lambda_{2,n}v_n\to 0\quad \textit{for some } \lambda_{1,n},\lambda_{2,n}\in\mathbb{R}, \tag{2.11}$$

$$I_\mu\left(u_n,v_n\right)\to c,\quad P\left(u_n,v_n\right)\to 0, \tag{2.12}$$

and

$$u_n^-,v_n^-\to 0\quad \textit{a.e. in } \mathbb{R}^N. \tag{2.13}$$

If

$$c\ne 0\quad \textit{and}\quad c<\frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}+\min\{0,m_\mu(a_1,0),m_\mu(0,a_2),m_\mu(a_1,a_2)\}, \tag{2.14}$$

then there exists $(u,v)\in H_r$ *with* $u,v>0$ *and* $\lambda_1,\lambda_2>0$ *such that, up to a subsequence,* $\left(u_n,v_n\right)\to(u,v)$ *in* H *and* $\left(\lambda_{1,n},\lambda_{2,n}\right)\to\left(\lambda_1,\lambda_2\right)$ *in* $\mathbb{R}^2$.

Proof. We divide the proof in three steps.

Step 1. We show that $\left\{\left(u_n,v_n\right)\right\}_{n\in\mathbb{N}}$ is bounded in H and $\lambda_{1,n},\lambda_{2,n}$ are bounded in $\mathbb{R}$.

If $q\in(2,2+\frac{4}{N})$, then $q\gamma_q<2$. Combining this with (2.1), and $P\left(u_n,v_n\right)\to 0$, for n large enough, we have

$$\begin{aligned}
c+1&\ge I_\mu\left(u_n,v_n\right)-\frac{1}{2^*}P\left(u_n,v_n\right)\\
&=\frac{1}{N}\left(\|\nabla u_n\|_2^2+\|\nabla v_n\|_2^2\right)+\mu\left(\frac{\gamma_q}{2^*}-\frac{1}{q}\right)\int_{\mathbb{R}^N}\left(|u_n|^q+|v_n|^q\right)\mathrm{d}x\\
&\ge\frac{1}{N}\left(\|\nabla u_n\|_2^2+\|\nabla v_n\|_2^2\right)-C\left(\|\nabla u_n\|_2^2+\|\nabla v_n\|_2^2\right)^{\frac{q\gamma_q}{2}}
\end{aligned}$$

for some $C>0$, which implies that $\left\{\left(u_n,v_n\right)\right\}_{n\in\mathbb{N}}$ is bounded in H. If $q\in[2+\frac{4}{N},2^*)$, we have $q\gamma_q\ge 2$, and using $P\left(u_n,v_n\right)\to 0$, we obtain for n large enough,

$$\begin{aligned}
c+1&\ge I_\mu\left(u_n,v_n\right)-\frac{1}{2}P\left(u_n,v_n\right)\\
&=\mu\left(\frac{\gamma_q}{2}-\frac{1}{q}\right)\int_{\mathbb{R}^N}|u_n|^q+|v_n|^q\,\mathrm{d}x+\frac{2}{N}\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\,\mathrm{d}x\\
&\ge C(\|\nabla u_n\|_2^2+\|\nabla v_n\|_2^2)+o_n(1)
\end{aligned}$$

for some $C>0$. This implies that $\left\{\left(u_n,v_n\right)\right\}_{n\in\mathbb{N}}$ is bounded in H. Moreover, by (2.11), we get that

$$\lambda_{1,n}=-\frac{1}{a_1^2}I'_\mu\left(u_n,v_n\right)\left[\left(u_n,0\right)\right]+o_n(1),$$

$$\lambda_{2,n}=-\frac{1}{a_2^2}I'_\mu\left(u_n,v_n\right)\left[\left(0,v_n\right)\right]+o_n(1).$$

Thus, $\lambda_{1,n},\lambda_{2,n}$ are bounded in $\mathbb{R}$. Hence, up to a subsequence, there exist $(u,v)\in H_r,\lambda_1,\lambda_2\in\mathbb{R}$ such that

$$\begin{cases}
\left(u_n,v_n\right)\rightharpoonup(u,v) & \text{in } H_r, L^{2^*}\left(\mathbb{R}^N\right)\times L^{2^*}\left(\mathbb{R}^N\right),\\
\left(u_n,v_n\right)\to(u,v) & \text{in } L^q\left(\mathbb{R}^N\right)\times L^q\left(\mathbb{R}^N\right)\ \text{for } q\in(2,2^*),\\
\left(u_n,v_n\right)\to(u,v) & \text{a.e. in } \mathbb{R}^N,\\
\left(\lambda_{1,n},\lambda_{2,n}\right)\to\left(\lambda_1,\lambda_2\right) & \text{in } \mathbb{R}^2.
\end{cases}$$

Furthermore, by (2.11) and (2.13), we infer that

$$\begin{cases} I'_\mu(u,v)+\lambda_1 u+\lambda_2 v=0, \\ u\geq 0, v\geq 0, \end{cases} \tag{2.15}$$

and so, $P(u,v)=0$.

Step 2. We prove that the weak limit satisfies $u\neq 0$ and $v\neq 0$, and so $u>0, v>0$ by the maximum principle. Arguing by contradiction, since there may be $u=0$ or $v=0$, we shall consider the following three cases.

Case 1. $u=0, v=0$.

Since $(u_n,v_n)\to(0,0)$ in $L^q(\mathbb{R}^N)\times L^q(\mathbb{R}^N)$, we have

$$\begin{aligned} 0&=P(u_n,v_n)+o_n(1)\\ &=\int_{\mathbb{R}^N}\left(|\nabla u_n|^2+|\nabla v_n|^2\right)\mathrm{d}x-2\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\,\mathrm{d}x+o_n(1). \end{aligned} \tag{2.16}$$

Without loss of generality, we may assume that

$$h_n=\int_{\mathbb{R}^N}\left(|\nabla u_n|^2+|\nabla v_n|^2\right)\mathrm{d}x\to h \quad\text{and}\quad w_n=2\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\,\mathrm{d}x\to w,$$

as $n\to\infty$. Passing to the limit in (2.16) as $n\to\infty$, using (2.5), we obtain

$$h=w\leq 2S_{\alpha,\beta}^{-\frac{2^*}{2}}h^{\frac{2^*}{2}}.$$

Therefore, either $h=0$ or $h\geq 2\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}$. If $h=0$, then this gives a contradiction to the fact that $c\neq 0$. So,

$$h\geq 2\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}. \tag{2.17}$$

Moreover, by (2.16) and (2.17), we derive

$$\begin{aligned} c&=\lim_{n\to\infty}I_\mu(u_n,v_n)\\ &=\lim_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2+|\nabla v_n|^2\right)\mathrm{d}x-\frac{2}{2^*}\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\,\mathrm{d}x\right)\\ &\geq\frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}, \end{aligned}$$

which contradicts (2.14).

Case 2. $u\neq 0, v=0$.

In this case, we have $u>0$ by the maximum principle and by (2.15), we have

$$\begin{cases} -\Delta u+\lambda_1 u=\mu u^{q-1}, & \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=a^2,\ u>0. \end{cases} \tag{2.18}$$

By using the papers of Li-Zou [43, Lemma 2.1] and Weinstein [44], we deduce that (2.18) has a unique positive solution with $a=\|u\|_2\leq a_1$. Thus,

$$m_\mu(a_1,0)\leq m_\mu(\|u\|_2,0)=I_\mu(u,0). \tag{2.19}$$

Let $\bar{u}_n=u_n-u$. Then by the Brézis-Lieb lemma [45] and Lemma 2.2, we have

$$\begin{aligned} 0&=P(u_n,v_n)+o_n(1)=P(\bar{u}_n,v_n)+P(u,0)+o_n(1)\\ &=\int_{\mathbb{R}^N}\left(|\nabla\bar{u}_n|^2+|\nabla v_n|^2\right)\mathrm{d}x-2\int_{\mathbb{R}^N}|\bar{u}_n|^\alpha|v_n|^\beta\,\mathrm{d}x+o_n(1). \end{aligned} \tag{2.20}$$

Without loss of generality, we may assume that

$$\ell_n=\int_{\mathbb{R}^N}\left(|\nabla\bar{u}_n|^2+|\nabla v_n|^2\right)\mathrm{d}x\to\ell \quad\text{and}\quad b_n=2\int_{\mathbb{R}^N}|\bar{u}_n|^\alpha|v_n|^\beta\,\mathrm{d}x\to b, \tag{2.21}$$

as $n\to\infty$. By (2.20), passing to the limit as $n\to\infty$, and (2.5), it follows that

$$\ell=b\leq 2S_{\alpha,\beta}^{-\frac{2^*}{2}}\ell^{\frac{2^*}{2}}.$$

Therefore, either $\ell=0$ or $\ell\geq 2\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}$. If $\ell=0$, then it contradicts to the fact that $c\neq 0$. So,

$$\ell\geq 2\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}. \tag{2.22}$$

Moreover, by virtue of (2.19)–(2.22), we derive

$$\begin{aligned} c &= \lim_{n\to\infty} I_\mu\left(u_n, v_n\right) = \lim_{n\to\infty} I_\mu\left(\bar{u}_n, v_n\right) + I_\mu(u,0) \\ &\geq \lim_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla\bar{u}_n|^2 + |\nabla v_n|^2\right)\mathrm{d}x - \frac{2}{2^*}\int_{\mathbb{R}^N}|\bar{u}_n|^\alpha |v_n|^\beta\,\mathrm{d}x\right) + m_\mu(a_1,0) \\ &\geq \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}} + m_\mu(a_1,0), \end{aligned}$$

which is a contradiction to (2.14).

Case 3. $u = 0, v \neq 0$.

Analogous to the proof in Case 2, we have

$$c \geq \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}} + m_\mu(0,a_2),$$

which is a contradiction to (2.14).

Step 3. $\left(u_n, v_n\right) \to (u,v)$ in H.

Let $(\bar{u}_n, \bar{v}_n) = (u_n - u, v_n - v)$. Then by the Brézis-Lieb lemma [45] and Lemma 2.2, we deduce that

$$\begin{aligned} 0 &= P\left(u_n, v_n\right) + o_n(1) = P\left(\bar{u}_n, \bar{v}_n\right) + P(u,v) + o_n(1) \\ &= \int_{\mathbb{R}^N}\left(|\nabla\bar{u}_n|^2 + |\nabla\bar{v}_n|^2\right)\mathrm{d}x - 2\int_{\mathbb{R}^N}|\bar{u}_n|^\alpha |\bar{v}_n|^\beta\,\mathrm{d}x + o_n(1). \end{aligned} \tag{2.23}$$

Without loss of generality, we may assume

$$\ell_n' = \int_{\mathbb{R}^N}\left(|\nabla\bar{u}_n|^2 + |\nabla\bar{v}_n|^2\right)\mathrm{d}x \to \ell' \quad \text{and} \quad b_n' = 2\int_{\mathbb{R}^N}|\bar{u}_n|^\alpha |\bar{v}_n|^\beta\,\mathrm{d}x \to b',$$

as $n\to\infty$. By using (2.23) and passing to the limit as $n\to\infty$ along with (2.5), we derive that

$$\ell' = b' \leq 2S_{\alpha,\beta}^{-\frac{2^*}{2}}\ell'^{\frac{2^*}{2}}.$$

Thus, either $\ell' = 0$ or $\ell' \geq 2\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}$. If $\ell' \geq 2\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}$ holds, then we deduce

$$\begin{aligned} c &= \lim_{n\to\infty} I_\mu\left(u_n, v_n\right) = \lim_{n\to\infty} I_\mu\left(\bar{u}_n, \bar{v}_n\right) + I_\mu(u,v) \\ &\geq \lim_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^N}|\left(|\nabla\bar{u}_n|^2 + |\nabla\bar{v}_n|^2\right)\mathrm{d}x - \frac{2}{2^*}\int_{\mathbb{R}^N}|\bar{u}_n|^\alpha |\bar{v}_n|^\beta\,\mathrm{d}x\right) \\ &\quad + m_\mu(\|u\|_2, \|v\|_2) \\ &\geq \lim_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla\bar{u}_n|^2 + |\nabla\bar{v}_n|^2\right)\mathrm{d}x - \frac{2}{2^*}\int_{\mathbb{R}^N}|\bar{u}_n|^\alpha |\bar{v}_n|^\beta\,\mathrm{d}x\right) + m_\mu(a_1,a_2) \\ &\geq \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}} + m_\mu(a_1,a_2), \end{aligned}$$

which is in contradiction to (2.14), where we have used $0 < \|u\|_2 \leq a_1$ and $0 < \|v\|_2 \leq a_2$ and assumption (2.10). Therefore, we must have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla\bar{u}_n|^2 + |\nabla\bar{v}_n|^2\right)\mathrm{d}x = 0.$$

Next, we claim that $\lambda_1, \lambda_2 > 0$. Indeed, if $\lambda_1 \leq 0$, then

$$-\Delta u = |\lambda_1| u + \mu u^{q-1} + \frac{2\alpha}{\alpha+\beta}u^{\alpha-1}v^\beta \geq 0 \quad \text{in } \mathbb{R}^N.$$

Then we can apply Lemma A.2 of Ikoma [46] deducing that $u = 0$ which is also a contradiction. Consequently, $\lambda_1 > 0$, and analogously $\lambda_2 > 0$, as claimed. Combining (2.11), (2.12), (2.15) and $P(u,v) = 0$, one has

$$\lambda_1 a_1^2 + \lambda_2 a_2^2 = \lambda_1\|u\|_2^2 + \lambda_2\|v\|_2^2.$$

It follows that $\|u\|_2 = a_1, \|v\|_2 = a_2$ and hence $\left(u_n, v_n\right) \to (u,v)$ in H. □

3. Proof of Theorem 1.1

In the L^2-subcritical case $2 < q < \bar{q} := 2 + \frac{4}{N}$, we have $0 < q\gamma_q < 2$. To begin our argument, we first introduce the following two positive constants

$$\mu_1 := \frac{2^{\frac{2q\gamma_q - 2 - 2^*}{2^*-2}}(2^*-2)\left(2^* - q\gamma_q\right)^{\frac{q\gamma_q - 2^*}{2^*-2}} q\left(2^* S_{\alpha,\beta}^{\frac{2^*}{2}}(2-q\gamma_q)\right)^{\frac{2-q\gamma_q}{2^*-2}}}{C_{N,q}(a_1^{q(1-\gamma_q)} + a_2^{q(1-\gamma_q)})}, \tag{3.1}$$

and

$$\mu_2 := \frac{(2^*-2)\left((2-q\gamma_q)S_{\alpha,\beta}^{\frac{2^*}{2}}\right)^{\frac{2-q\gamma_q}{2^*-2}}}{\gamma_q C_{N,q}(a_1^{q(1-\gamma_q)}+a_2^{q(1-\gamma_q)})\,(2^*-q\gamma_q)^{\frac{2^*-q\gamma_q}{2^*-2}}\,2^{\frac{2-q\gamma_q}{2^*-2}}}. \tag{3.2}$$

We consider the constrained functional $I_\mu|_{S_{a_1}\times S_{a_2}}$. For any $(u,v)\in S_{a_1}\times S_{a_2}$, by account of (2.1) and (2.5), we have that

$$\begin{aligned} I_\mu(u,v) &\geq \frac{1}{2}\left(\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)\mathrm{d}x\right) \\ &\quad -\frac{2S_{\alpha,\beta}^{-\frac{2^*}{2}}}{2^*}\left(\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)\mathrm{d}x\right)^{\frac{2^*}{2}} \\ &\quad -\frac{\mu}{q}C_{N,q}a_1^{q(1-\gamma_q)}\|\nabla u\|_2^{q\gamma_q}-\frac{\mu}{q}C_{N,q}a_2^{q(1-\gamma_q)}\|\nabla v\|_2^{q\gamma_q} \\ &\geq h\left(\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)\mathrm{d}x\right)^{\frac{1}{2}}, \end{aligned} \tag{3.3}$$

where the function $h:\mathbb{R}^+\to\mathbb{R}$ is defined by

$$h(t)=\frac{1}{2}t^2-\frac{2S_{\alpha,\beta}^{-\frac{2^*}{2}}}{2^*}t^{2^*}-\frac{\mu}{q}C_{N,q}(a_1^{q(1-\gamma_q)}+a_2^{q(1-\gamma_q)})t^{q\gamma_q}.$$

From $q\gamma_q<2$, we get that $h(0^+)=0^-$ and $h(+\infty)=-\infty$.

Lemma 3.1. *Assume that $0<\mu<\mu_1$. Then the function $h(\cdot)$ has exactly two critical points, one is a local strict minimum at negative level, the other one is a global maximum at positive level. Furthermore, there exist $0<R_0<R_1$, such that $h(R_0)=h(R_1)=0$ and $h(t)>0$ if and only if $t\in(R_0,R_1)$.*

Proof. For $t>0$, we have $h(t)>0$ if and only if

$$\varphi(t)>\frac{\mu}{q}C_{N,q}\left(a_1^{q(1-\gamma_q)}+a_2^{q(1-\gamma_q)}\right)\quad\text{with}\quad\varphi(t)=\frac{1}{2}t^{2-q\gamma_q}-\frac{2S_{\alpha,\beta}^{-\frac{2^*}{2}}}{2^*}t^{2^*-q\gamma_q}.$$

In view of

$$\varphi'(t)=\frac{2-q\gamma_q}{2}t^{1-q\gamma_q}-\frac{2S_{\alpha,\beta}^{-\frac{2^*}{2}}}{2^*}(2^*-q\gamma_q)t^{2^*-q\gamma_q-1},$$

it is not difficult to check that $\varphi(\cdot)$ has a unique critical point at

$$\bar{t}=\left(\frac{2^*(2-q\gamma_q)}{4(2^*-q\gamma_q)}S_{\alpha,\beta}^{\frac{2^*}{2}}\right)^{\frac{1}{2^*-2}}, \tag{3.4}$$

and $\varphi(\cdot)$ is increasing on $(0,\bar{t})$ and decreasing on $(\bar{t},+\infty)$. Moreover, the maximum level is

$$\varphi(\bar{t})=2^{\frac{2q\gamma_q-2-2^*}{2^*-2}}(2^*-2)\left(2^*-q\gamma_q\right)^{\frac{q\gamma_q-2^*}{2^*-2}}\left(2^*S_{\alpha,\beta}^{\frac{2^*}{2}}(2-q\gamma_q)\right)^{\frac{2-q\gamma_q}{2^*-2}}.$$

Thus, there exist $0<R_0<R_1$ such that $h(R_0)=h(R_1)=0$ and $h(t)>0$ if and only if $t\in(R_0,R_1)$. Moreover, h is positive on an open interval (R_0,R_1) if and only if $\varphi(\bar{t})>\frac{\mu}{q}C_{N,q}(a_1^{q(1-\gamma_q)}+a_2^{q(1-\gamma_q)})$, that is, $\mu<\mu_1$ holds. Since $h(0^+)=0^-$, $h(+\infty)=-\infty$ and h is positive on an open interval (R_0,R_1), it is easy to see that h has a global maximum at positive level in (R_0,R_1) as well as a local minimum point at negative level in $(0,R_0)$. Note that

$$h'(t)=t^{q\gamma_q-1}\left[t^{2-q\gamma_q}-2S_{\alpha,\beta}^{-\frac{2^*}{2}}t^{2^*-q\gamma_q}-\mu\gamma_qC_{N,q}(a_1^{q(1-\gamma_q)}+a_2^{q(1-\gamma_q)})\right]=0$$

if and only if

$$\psi(t)=\mu\gamma_qC_{N,q}\left(a_1^{q(1-\gamma_q)}+a_2^{q(1-\gamma_q)}\right)\quad\text{with}\quad\psi(t)=t^{2-q\gamma_q}-2S_{\alpha,\beta}^{-\frac{2^*}{2}}t^{2^*-q\gamma_q}.$$

Clearly, $\psi(\cdot)$ has only one critical point $\hat{t}$, which is a strict maximum. Therefore, the above equation has at most two solutions, which implies that h only has a local strict minimum at negative level and a global strict maximum at positive level and no other critical points. Thus, $h(\cdot)$ has exactly two critical points $0<t_1<\hat{t}<t_2$ with

$$h(t_1)=\min_{0<t<\hat{t}}h(t)<0\quad\text{and}\quad h(t_2)=\max_{t>0}h(t)>0.\quad\square$$

Lemma 3.2. *Assume that* $0 < \mu < \mu_2$. *Then* $\mathcal{P}^0_{a_1,a_2} = \emptyset$ *and* $\mathcal{P}_{a_1,a_2}$ *is a smooth manifold of codimension 3 in* H.

Proof. We first prove that $\mathcal{P}^0_{a_1,a_2} = \emptyset$ implies that $\mathcal{P}_{a_1,a_2}$ is a smooth manifold of codimension 3 in H. We note that $\mathcal{P}_{a_1,a_2}$ is defined by $P(u,v) = 0, G(u) = 0, F(v) = 0$, where

$$G(u) = a_1^2 - \int_{\mathbb{R}^N} u^2\,\mathrm{d}x, \quad F(v) = a_2^2 - \int_{\mathbb{R}^N} v^2\,\mathrm{d}x.$$

It suffices to show that the differential

$$d(P,G,F) : H \to \mathbb{R}^3$$

is surjective. Assuming that it is not true, there must be that $dP(u,v)$ is a linear combination of $dG(u)$ and $dF(v)$ according to the independence of $dG(u)$ and $dF(v)$. That is, there exist $\nu_1, \nu_2 \in \mathbb{R}$ such that (u,v) is a weak solution of the system

$$\begin{cases} -\Delta u + \nu_1 u = \dfrac{q\gamma_q}{2}\mu|u|^{q-2}u + \alpha|u|^{\alpha-2}u|v|^\beta & \text{in } \mathbb{R}^N, \\ -\Delta v + \nu_2 v = \dfrac{q\gamma_q}{2}\mu|v|^{q-2}v + \beta|u|^{\alpha}|v|^{\beta-2}v & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x = a_1^2, \ \int_{\mathbb{R}^N} |v|^2\,\mathrm{d}x = a_2^2. \end{cases} \tag{3.5}$$

However, by the Pohozaev identity for (3.5), we have

$$2\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right)\,\mathrm{d}x = \mu q\gamma_q^2 \int_{\mathbb{R}^N} \left(|u|^q + |v|^q\right)\,\mathrm{d}x + 22^* \int_{\mathbb{R}^N} |u|^\alpha|v|^\beta\,\mathrm{d}x,$$

which implies that $(u,v) \in \mathcal{P}^0_{a_1,a_2}$, a contradiction.

Now, we prove that $\mathcal{P}^0_{a_1,a_2} = \emptyset$. Arguing by contradiction, there exists $(u,v) \in \mathcal{P}^0_{a_1,a_2}$. Let $\rho = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2)\,\mathrm{d}x\right)^{\frac{1}{2}}$ and let

$$\begin{aligned} W(t) :&= t\Phi'_{(u,v)}(0) - \Phi''_{(u,v)}(0) \\ &= (t-2)\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right)\,\mathrm{d}x - \mu\left(t - q\gamma_q\right)\gamma_q \int_{\mathbb{R}^N} \left(|u|^q + |v|^q\right)\,\mathrm{d}x \\ &\quad - 2\left(t - 2^*\right)\int_{\mathbb{R}^N} |u|^\alpha|v|^\beta\,\mathrm{d}x \\ &= 0. \end{aligned}$$

In view of $W(q\gamma_q) = 0$ and (2.5), we obtain

$$(2 - q\gamma_q)\rho^2 = 2\left(2^* - q\gamma_q\right)\int_{\mathbb{R}^N} |u|^\alpha|v|^\beta\,\mathrm{d}x \le 2\left(2^* - q\gamma_q\right) S_{\alpha,\beta}^{-\frac{2^*}{2}} \rho^{2^*}.$$

It follows from $q\gamma_q < 2$ that

$$\rho \ge \left(\frac{2 - q\gamma_q}{2\left(2^* - q\gamma_q\right)} S_{\alpha,\beta}^{\frac{2^*}{2}}\right)^{\frac{1}{2^*-2}}. \tag{3.6}$$

Moreover, combining (3.6) with $W(2^*) = 0$, we infer to

$$\begin{aligned} (2^* - 2) &= \left(2^* - q\gamma_q\right)\gamma_q\mu\rho^{-2}\int_{\mathbb{R}^N} |u|^q + |v|^q\,\mathrm{d}x \\ &\le \left(2^* - q\gamma_q\right)\gamma_q\mu C_{N,q}(a_1^{q(1-\gamma_q)} + a_2^{q(1-\gamma_q)})\rho^{q\gamma_q - 2} \\ &\le \gamma_q\mu C_{N,q}(a_1^{q(1-\gamma_q)} + a_2^{q(1-\gamma_q)})\left(2^* - q\gamma_q\right)^{\frac{2^*-q\gamma_q}{2^*-2}} \left(\frac{2S_{\alpha,\beta}^{-\frac{2^*}{2}}}{2 - q\gamma_q}\right)^{\frac{2-q\gamma_q}{2^*-2}}, \end{aligned}$$

that is

$$\mu \ge \mu_2 := \frac{(2^* - 2)\left((2 - q\gamma_q)S_{\alpha,\beta}^{\frac{2^*}{2}}\right)^{\frac{2-q\gamma_q}{2^*-2}}}{\gamma_q C_{N,q}(a_1^{q(1-\gamma_q)} + a_2^{q(1-\gamma_q)})\left(2^* - q\gamma_q\right)^{\frac{2^*-q\gamma_q}{2^*-2}} 2^{\frac{2-q\gamma_q}{2^*-2}}},$$

which leads to a contradiction to our assumptions. Hence, $\mathcal{P}^0_{a_1,a_2} = \emptyset$. □

By using Lemmas 3.1 and 3.2, we can describe the geometry of $\mathcal{P}_{a_1,a_2}$. The manifold $\mathcal{P}_{a_1,a_2}$ is then divided into its two components $\mathcal{P}^+_{a_1,a_2}$ and $\mathcal{P}^-_{a_1,a_2}$ having disjoint closure.

Lemma 3.3. *For every* $(u,v) \in S_{a_1} \times S_{a_2}$*, the function* $\Phi_{(u,v)}(\cdot)$ *has exactly two critical points* $s_{(u,v)} < t_{(u,v)}$ *and two zero points* $c_{(u,v)} < d_{(u,v)}$ *with* $s_{(u,v)} < c_{(u,v)} < t_{(u,v)} < d_{(u,v)}$*. Moreover, it holds:*

(i) $s \star (u,v) \in \mathcal{P}^+_{a_1,a_2}$ *if and only if* $s = s_{(u,v)}$;
$s \star (u,v) \in \mathcal{P}^-_{a_1,a_2}$ *if and only if* $s = t_{(u,v)}$;

(ii) $(\|\nabla(s\star u)\|_2^2 + \|\nabla(s\star v)\|_2^2)^{1/2} \le R_0$ *for each* $s \le c_{(u,v)}$ *and*

$$\begin{aligned} &I_\mu(s_{(u,v)} \star (u,v)) \\ &= \min\{I_\mu(s\star(u,v)) : s \in \mathbb{R} \text{ and } (\|\nabla(s\star u)\|_2^2 + \|\nabla(s\star v)\|_2^2)^{1/2} \le R_0\} < 0; \end{aligned}$$

(iii) $I_\mu(t_{(u,v)} \star (u,v)) = \max_{s\in\mathbb{R}} I_\mu(s\star(u,v)) > 0$ *and* $\Phi_{(u,v)}(s)$ *is strictly decreasing on* $(t_{(u,v)}, +\infty)$;

(iv) *The maps* $(u,v) \mapsto s_{(u,v)} \in \mathbb{R}$ *and* $(u,v) \mapsto t_{(u,v)} \in \mathbb{R}$ *are of class* C^1.

Proof. Let $(u,v) \in S_{a_1} \times S_{a_2}$. First, we show that $\Phi_{(u,v)}(\cdot)$ has at least two critical points. We recall that by (3.3), we obtain

$$\Phi_{(u,v)}(s) = I_\mu(s\star(u,v)) \ge h\left(e^s(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{1/2}\right),$$

and so

$$\Phi_{(u,v)}(s) > 0 \quad \text{for all } s \in \left(\log\frac{R_0}{(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{1/2}},\ \log\frac{R_1}{(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{1/2}}\right).$$

Clearly, $\Phi_{(u,v)}(-\infty) = 0^-$ and $\Phi_{(u,v)}(+\infty) = -\infty$. It is proved that $\Phi_{(u,v)}(\cdot)$ has at least two critical points $s_{(u,v)} < t_{(u,v)}$, where $t_{(u,v)}$ is the global maximum point at the positive level, and $s_{(u,v)}$ is the local minimum point on

$$\left(-\infty, \log\frac{R_0}{(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{1/2}}\right)$$

at the negative level. Now we claim that $\Phi_{(u,v)}(\cdot)$ has no other critical points. In fact, as $\Phi'_{(u,v)}(s) = 0$, we get

$$g(s) = \gamma_q \mu \int_{\mathbb{R}^N} \left(|u|^q + |v|^q\right) \mathrm{d}x$$

with

$$g(s) := e^{(2-q\gamma_q)s}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + |\nabla v|^2\right)\mathrm{d}x - 2e^{(2^*-q\gamma_q)s}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x.$$

It can be seen that $g(\cdot)$ has a unique maximum point, so the above equation has at most two solutions.

On account of $\Phi'_{(u,v)}(s) = P(s\star(u,v))$, it gives $s\star(u,v) \in \mathcal{P}_{a_1,a_2}$ implying that $s = s_{(u,v)}$ or $t_{(u,v)}$. By combining that $s_{(u,v)}$ is a local minimum point of $\Phi_{(u,v)}(s)$ and $\mathcal{P}^0_{a_1,a_2} = \emptyset$, we immediately infer that $\Phi''_{(u,v)}(s_{(u,v)}) > 0$, which implies that $s_{(u,v)} \star (u,v) \in \mathcal{P}^+_{a_1,a_2}$. Similarly, we have that $t_{(u,v)} \star (u,v) \in \mathcal{P}^-_{a_1,a_2}$.

By the monotonicity and recalling the behavior at infinity of $\Phi_{(u,v)}(\cdot)$, we see that $\Phi_{(u,v)}(\cdot)$ has exactly two zero points $c_{(u,v)}$ and $d_{(u,v)}$ with $s_{(u,v)} < c_{(u,v)} < t_{(u,v)} < d_{(u,v)}$. Particularly, $\Phi_{(u,v)}(\cdot)$ is strictly decreasing on $(t_u, +\infty)$.

Finally, we show that $(u,v) \mapsto s_{(u,v)} \in \mathbb{R}$ and $(u,v) \mapsto t_{(u,v)} \in \mathbb{R}$ are of class C^1. Indeed, we can apply the implicit function theorem on $\Psi(s,u,v) := \Phi'_{(u,v)}(s)$. Then we have

$$\begin{aligned} &\Psi(s_{(u,v)}, u, v) = \Psi(t_{(u,v)}, u, v) = 0, \\ &\partial_s\Psi(s_{(u,v)}, u, v) = \Phi''_{(u,v)}(s_{(u,v)}) > 0, \\ &\partial_s\Psi(t_{(u,v)}, u, v) = \Phi''_{(u,v)}(t_{(u,v)}) < 0 \end{aligned}$$

and $\mathcal{P}^0_{a_1,a_2} = \emptyset$ imply that it is not possible to pass with continuity from $\mathcal{P}^+_{a_1,a_2}$ to $\mathcal{P}^-_{a_1,a_2}$. Thus, we know that $(u,v) \mapsto s_{(u,v)} \in \mathbb{R}$ and $(u,v) \mapsto t_{(u,v)} \in \mathbb{R}$ are of class C^1. □

By using Lemma 2.1 of Li-Zou [43], if $q \in (2, 2+\frac{4}{N})$, it holds

$$m_\mu(a_1, 0) < 0 \quad \text{and} \quad m_\mu(0, a_2) < 0. \tag{3.7}$$

For $k > 0$, we define

$$A_k = \left\{(u,v) \in S_{a_1} \times S_{a_2} : \left(\int_{\mathbb{R}^N}\left(|\nabla u|^2 + |\nabla v|^2\right)\mathrm{d}x\right)^{1/2} < k\right\}.$$

Lemma 3.4. *If* $0 < \mu < \min\{\mu_1, \mu_2\}$*, then the following statements hold.*

(i) $m_\mu(a_1, a_2) = \inf_{A_{R_0}} I_\mu(u,v) < 0$;

(ii) $m_\mu(a_1, a_2) \le m_\mu\left(b_1, b_2\right)$ *for any* $0 < b_1 \le a_1, 0 < b_2 \le a_2$.

Proof. (i) By Lemma 3.3, we have

$$\mathcal{P}^{+}_{a_1,a_2} = \{s_{(u,v)} \star (u,v) : (u,v) \in S_{a_1} \times S_{a_2}\} \subset A_{R_0}$$

and

$$m_\mu(a_1,a_2) = \inf_{\mathcal{P}_{a_1,a_2}} I_\mu(u,v) = \inf_{\mathcal{P}^{+}_{a_1,a_2}} I_\mu(u,v) < 0.$$

Clearly, $m_\mu(a_1,a_2) \geq \inf_{A_{R_0}} I_\mu(u,v)$. On the other hand, for any $(u,v) \in A_{R_0}$, we get

$$m_\mu(a_1,a_2) \leq I_\mu(s_{(u,v)} \star (u,v)) \leq I_\mu(u,v),$$

which implies that $m_\mu(a_1,a_2) \leq \inf_{A_{R_0}} I_\mu(u,v)$. Thus, it holds $m_\mu(a_1,a_2) = \inf_{A_{R_0}} I_\mu(u,v)$.

(ii) This can be proved by following the strategy by Jeanjean-Lu [31, Lemma 3.2], where a scalar equation is considered, with minor modifications. By the definition of $\bar{t}$ in (3.4), we have as in (i), that

$$m_\mu(a_1,a_2) = \inf_{A_{\bar{t}}} I_\mu(u,v).$$

For arbitrary $\varepsilon > 0$, we prove that $m_\mu(a_1,a_2) \leq m_\mu\left(b_1,b_2\right) + \varepsilon$. Let $(u,v) \in A_{\bar{t}}$ be such that

$$I_\mu(u,v) \leq m_\mu\left(b_1,b_2\right) + \frac{\varepsilon}{2}, \tag{3.8}$$

and let $\phi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function with $\phi \in [0,1]$, $\phi \equiv 1$ on $B_1(0)$ and $\phi \equiv 0$ on $\mathbb{R}^3 \setminus B_2(0)$. For $\delta > 0$, we consider $u_\delta(x) := u(x)\phi(\delta x)$ and $v_\delta(x) := v(x)\phi(\delta x)$. Obviously, $(u_\delta, v_\delta) \to (u,v)$ in H as $\delta \to 0$. As a consequence, for $\eta > 0$ small enough, there exists $\delta > 0$ small enough such that

$$I_\mu(u_\delta,v_\delta) \leq I_\mu(u,v) + \frac{\varepsilon}{4} \quad \text{and} \quad \left(\int_{\mathbb{R}^N} \left(|\nabla u_\delta|^2 + |\nabla v_\delta|^2\right) \mathrm{d}x\right)^{1/2} < \bar{t} - \eta.$$

Now, we take $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\mathrm{supp}(\varphi) \subset B(0, 1+\frac{4}{\delta}) \setminus B(0, \frac{4}{\delta})$. Define

$$w_a = \frac{\sqrt{a_1^2 - \|u_\delta\|_2^2}}{\|\varphi\|_2} \times \varphi \quad \text{and} \quad w_b = \frac{\sqrt{a_2^2 - \|v_\delta\|_2^2}}{\|\varphi\|_2} \times \varphi,$$

then

$$\left(\mathrm{supp}\left(u_\delta\right) \cup \mathrm{supp}\left(v_\delta\right)\right) \cap \left(\mathrm{supp}\left(s \star w_a\right) \cup \mathrm{supp}\left(s \star w_b\right)\right) = \emptyset,$$

for $s < 0$. Hence $\left(u_\delta + s \star w_a, v_\delta + s \star w_b\right) \in S_{a_1} \times S_{a_2}$. Note that

$$I_\mu\left(s \star \left(w_a, w_b\right)\right) \to 0 \quad \text{and} \quad \left(\int_{\mathbb{R}^N} \left(|\nabla(s \star w_a)|^2 + |\nabla(s \star w_b)|^2\right) \mathrm{d}x\right)^{1/2} \to 0$$

as $s \to -\infty$, thus we obtain

$$\begin{aligned} I_\mu\left(s \star \left(w_a, w_b\right)\right) &\leq \frac{\varepsilon}{4}, \\ \left(\int_{\mathbb{R}^N} \left(|\nabla(s \star w_a)|^2 + |\nabla(s \star w_b)|^2\right) \mathrm{d}x\right)^{1/2} &\leq \frac{\eta}{2} \end{aligned} \tag{3.9}$$

for $s < 0$ sufficiently close to $-\infty$. Consequently,

$$\left(\int_{\mathbb{R}^N} \left(|\nabla(u_\delta + s \star w_a)|^2 + |\nabla(v_\delta + s \star w_b)|^2\right) \mathrm{d}x\right)^{1/2} < \bar{t},$$

and combining (3.8)–(3.9), we have

$$\begin{aligned} m_\mu(a_1,a_2) &\leq I_\mu\left(u_\delta + s \star w_a, v_\delta + s \star w_b\right) = I_\mu\left(u_\delta, v_\delta\right) + I_\mu\left(s \star w_a, s \star w_b\right) \\ &\leq m_\mu\left(b_1,b_2\right) + \varepsilon, \end{aligned}$$

which completes the proof. □

Proof of Theorem 1.1. By choosing $0 < \mu < \min\{\mu_1,\mu_2\}$, then combining (3.7), Lemmas 2.5 and 3.4, it is sufficient to show that at the $m_\mu(a_1,a_2)$ level, there exists a radially symmetric Palais–Smale sequence for $I_\mu|_{S_{a_1} \times S_{a_2}}$ such that $P\left(u_n,v_n\right) \to 0$ and $u_n^-, v_n^- \to 0$ a.e. in $\mathbb{R}^N$.

Let $m_r(a_1,a_2) = \inf_{A_{R_0} \cap H_r} I_\mu(u,v)$. By the symmetric decreasing rearrangement, it is easy to verify that $m_\mu(a_1,a_2) = m_r(a_1,a_2)$. Taking a minimizing sequence $\{(\tilde{u}_n,\tilde{v}_n)\}_{n\in\mathbb{N}}$ for $m_\mu(a_1,a_2) = \inf_{A_{R_0} \cap H_r} I_\mu(u,v)$, after passing to $(|\tilde{u}_n|, |\tilde{v}_n|)$ we may assume that $(\tilde{u}_n,\tilde{v}_n)$ are nonnegative. Furthermore, by using $I_\mu(s_{(\tilde{u}_n,\tilde{v}_n)} \star (\tilde{u}_n,\tilde{v}_n)) \leq I_\mu(\tilde{u}_n,\tilde{v}_n)$, and replacing $(\tilde{u}_n,\tilde{v}_n)$ by $(\hat{u}_n,\hat{v}_n) := s_{(\tilde{u}_n,\tilde{v}_n)} \star (\tilde{u}_n,\tilde{v}_n)$, we get a minimizing sequence $(\hat{u}_n,\hat{v}_n) \in \mathcal{P}^{+}_{a_1,a_2,r}$. Thus, by Ekeland's variational principle (see, for example, Willem [36]), there exists a radially symmetric Palais–Smale sequence (u_n, v_n) for $I_\mu|_{S_{a_1,r}\times S_{a_2,r}}$ (hence a Palais–Smale sequence for $I_\mu|_{S_{a_1}\times S_{a_2}}$) satisfying $\|(u_n, v_n) - (\hat{u}_n, \hat{v}_n)\| \to 0$ as $n \to \infty$, which indicates that

$$P(u_n, v_n) = P(\hat{u}_n, \hat{v}_n) + o_n(1) \to 0 \quad \text{and} \quad u_n^-, v_n^- \to 0 \quad \text{a.e. in } \mathbb{R}^N.$$

Then Lemma 2.5 with $c = m_r(a_1, a_2)$ implies that there exists a $(u, v) \in H_r, u, v > 0$ and $\lambda_1, \lambda_2 > 0$ such that, up to a subsequence if necessary, $(u_n, v_n) \to (u, v)$ in H_r and $(\lambda_{1,n}, \lambda_{2,n}) \to (\lambda_1, \lambda_2)$ in $(\mathbb{R}^+)^2$. From the strong convergence, $(u, v) \in \mathcal{P}_{a_1,a_2}$ is a solution of (1.1)–(1.2) and thus a normalized ground state. □

4. Proof of Theorem 1.2

In this section, we deal with the L^2-supercritical case $\bar{q} := 2 + \frac{4}{N} < q < 2^*$. We consider once again the Pohozaev manifold $\mathcal{P}_{a_1,a_2}$, which can be decomposed as

$$\mathcal{P}_{a_1,a_2} = \mathcal{P}^+_{a_1,a_2} \cup \mathcal{P}^0_{a_1,a_2} \cup \mathcal{P}^-_{a_1,a_2}.$$

If there exists $(u, v) \in \mathcal{P}^0_{a_1,a_2}$, then we have that

$$(q\gamma_q - 2)\mu\gamma_q \int_{\mathbb{R}^N} \left(|u|^q + |v|^q\right) \mathrm{d}x + 2(2^* - 2) \int_{\mathbb{R}^N} |u|^\alpha\, |v|^\beta \,\mathrm{d}x = 0.$$

Since $q\gamma_q > 2$, there must be $(u, v) = (0, 0)$, which contradicts the fact that $(u, v) \in S_{a_1} \times S_{a_2}$. This implies that $\mathcal{P}^0_{a_1,a_2} = \emptyset$ and then, as in Lemma 3.2, we can prove that $\mathcal{P}_{a_1,a_2}$ is a smooth manifold of codimension 3 in H. However, we know that the geometry of $\mathcal{P}_{a_1,a_2}$ will be different from the one in Lemma 3.3.

Lemma 4.1. *For each $(u, v) \in S_{a_1} \times S_{a_2}$, there exists a unique $t_{(u,v)} \in \mathbb{R}$ such that $t_{(u,v)} \star (u, v) \in \mathcal{P}_{a_1,a_2}$, where $t_{(u,v)}$ is the unique critical point of the function of $\Phi_{(u,v)}$ and it is a strict maximum point at positive level. Moreover, it holds:*

(i) $\mathcal{P}_{a_1,a_2} = \mathcal{P}^-_{a_1,a_2}$;
(ii) *$\Phi_{(u,v)}(s)$ is strictly increasing on $(-\infty, t_{(u,v)})$ and*

$$\Phi_{(u,v)}(t_{(u,v)}) = \max_{s\in\mathbb{R}} \Phi_{(u,v)}(s) > 0;$$

(iii) *The map $(u, v) \mapsto t_{(u,v)} \in \mathbb{R}$ is of class C^1;*
(iv) *$P(u, v) < 0$ if and only if $t_{(u,v)} < 0$.*

Proof. In view of

$$\begin{aligned}\Phi_{(u,v)}(s) &= I_\mu(s \star (u, v)) \\ &= \frac{e^{2s}}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) \mathrm{d}x - \frac{\mu e^{q\gamma_q s}}{q} \int_{\mathbb{R}^N} \left(|u|^q + |v|^q\right) \mathrm{d}x \\ &\quad - \frac{2e^{2^* s}}{2^*} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \,\mathrm{d}x,\end{aligned}$$

we have

$$\begin{aligned}\Phi'_{(u,v)}(s) = e^{2s} &\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) \mathrm{d}x - \mu\gamma_q e^{q\gamma_q s} \int_{\mathbb{R}^N} \left(|u|^q + |v|^q\right) \mathrm{d}x \\ &- 2e^{2^* s} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \,\mathrm{d}x.\end{aligned}$$

It is easy to see that $\Phi'_{(u,v)}(s) = 0$ if and only if

$$\begin{aligned}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) \mathrm{d}x = \mu\gamma_q e^{(q\gamma_q - 2)s} &\int_{\mathbb{R}^N} \left(|u|^q + |v|^q\right) \mathrm{d}x \\ + 2e^{(2^*-2)s} &\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \,\mathrm{d}x \triangleq g(s).\end{aligned}$$

Clearly, $g(\cdot)$ is positive, continuous and monotone increasing, and $g(s) \to 0^+$ as $s \to -\infty$ and $g(s) \to +\infty$ as $s \to +\infty$. Therefore, there exists a unique point $t_{(u,v)}$ such that $t_{(u,v)} \star (u, v) \in \mathcal{P}_{a_1,a_2}$, where $t_{(u,v)}$ is the unique critical point of $\Phi_{(u,v)}(\cdot)$ and it is a strict maximum point at positive level. By maximality, we have that $\Phi''_{(u,v)}(t_{(u,v)}) \le 0$ and since $\mathcal{P}^0_{a_1,a_2} = \emptyset$, we conclude that $t_{(u,v)} \star (u, v) \in \mathcal{P}^-_{a_1,a_2}$ and $\mathcal{P}_{a_1,a_2} = \mathcal{P}^-_{a_1,a_2}$ since $\Phi_{(u,v)}(\cdot)$ has exactly one inflection point. In order to show that the map $(u, v) \mapsto t_{(u,v)} \in \mathbb{R}$ is of class C^1, we can apply the implicit function theorem as in Lemma 3.3. Finally, since $\Phi'_{(u,v)}(s) < 0$ if and only if $s > t_{(u,v)}$, so $P(u, v) = \Phi'_{(u,v)}(0) < 0$ if and only if $t_{(u,v)} < 0$. □

Lemma 4.2. *The minimum $m_\mu(a_1, a_2)$ has a the following minimax representation*

$$m_\mu(a_1, a_2) = \inf_{S_{a_1}\times S_{a_2}} \max_{s\in\mathbb{R}} I_\mu(s \star (u, v)).$$

Proof. For $(u,v)\in S_{a_1}\times S_{a_2}$, by Lemma 4.1, we have

$$\max_{s\in\mathbb{R}} I_\mu(s\star(u,v)) = I_\mu(t_{(u,v)}\star(u,v)) \geq m_\mu(a_1,a_2).$$

Hence

$$m_\mu(a_1,a_2) \leq \inf_{S_{a_1}\times S_{a_2}} \max_{s\in\mathbb{R}} I_\mu(s\star(u,v)).$$

On the other hand, by using Lemma 4.1, we also obtain

$$I_\mu(u,v) = \max_{s\in\mathbb{R}} I_\mu(s\star(u,v)) \geq \inf_{S_{a_1}\times S_{a_2}} \max_{s\in\mathbb{R}} I_\mu(s\star(u,v)) \quad \text{if } (u,v)\in\mathcal{P}_{a_1,a_2},$$

and thus we conclude that

$$m_\mu(a_1,a_2) = \inf_{S_{a_1}\times S_{a_2}} \max_{s\in\mathbb{R}} I_\mu(s\star(u,v)). \quad \square$$

We recall the following useful lemma, which is needed in proving Lemma 4.4 below, see the paper of Bartsch-Soave [14].

Lemma 4.3. *The map $(s,u)\in\mathbb{R}\times H^1(\mathbb{R}^N)\to s\star u\in H^1(\mathbb{R}^N)$ is continuous.*

Now, we give a way to find the required Palais–Smale sequence in Lemma 2.5.

Lemma 4.4. *There exists a radial Palais–Smale sequence for $I_\mu|_{S_{a_1}\times S_{a_2}}$ at level $m_\mu(a_1,a_2)$ with $P\left(u_n,v_n\right)\to 0$ and $u_n^-, v_n^-\to 0$ a.e. in $\mathbb{R}^N$.*

Proof. We use the strategy firstly introduced in Jeanjean [24] and consider the functional $\widetilde{I}_\mu : \mathbb{R}\times H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)\to\mathbb{R}$ defined by

$$\widetilde{I}_\mu(s,u,v) := I_\mu(s\star(u,v))$$

on the constraint $\mathbb{R}\times S_{a_1,r}\times S_{a_2,r}$. It is straightforward to check that $\widetilde{I}_\mu$ is of class C^1. Let

$$I_\mu^c := \{(u,v)\in S_{a_1}\times S_{a_2} : I_\mu(u,v)\leq c\}.$$

Note that, for any $(u,v)\in S_{a_1}\times S_{a_2}$, we have

$$\begin{aligned} I_\mu(u,v) \geq{}& \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)\,\mathrm{d}x - \frac{2S_{\alpha,\beta}^{-\frac{2^*}{2}}}{2^*}\left(\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)\,\mathrm{d}x\right)^{\frac{2^*}{2}} \\ &- \frac{\mu}{q}C_{N,q}a_1^{q(1-\gamma_q)}\|\nabla u\|_2^{q\gamma_q} - \frac{\mu}{q}C_{N,q}a_2^{q(1-\gamma_q)}\|\nabla v\|_2^{q\gamma_q} > 0 \end{aligned}$$

and

$$\begin{aligned} P(u,v) \geq{}& \int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)\,\mathrm{d}x - 2S_{\alpha,\beta}^{-\frac{2^*}{2}}\left(\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)\,\mathrm{d}x\right)^{\frac{2^*}{2}} \\ &- \mu\gamma_q C_{N,q}a_1^{q(1-\gamma_q)}\|\nabla u\|_2^{q\gamma_q} - \mu\gamma_q C_{N,q}a_2^{q(1-\gamma_q)}\|\nabla v\|_2^{q\gamma_q} > 0, \end{aligned}$$

if $(u,v)\in\overline{A}_k$ with k small enough. By Lemma 4.1, we know that $m_\mu(a_1,a_2)>0$, thus if necessary replacing k by a smaller quantity, we also have

$$I_\mu(u,v) \leq \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)\,\mathrm{d}x < m_\mu(a_1,a_2).$$

We consider now the following minimax level

$$\sigma := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\widetilde{I}_\mu(\gamma(t))$$

with

$$\begin{aligned} \Gamma = \Big\{&\gamma = \left(\alpha,\varphi_1,\varphi_2\right)\in C\left([0,1],\mathbb{R}\times S_{a_1,r}\times S_{a_2,r}\right) : \\ &\gamma(0)\in\{0\}\times\overline{A}_k, \gamma(1)\in\{0\}\times I_\mu^0\Big\}. \end{aligned}$$

Next, we shall show that $\sigma = m_\mu(a_1,a_2)$. On the one hand, we note that for any $(u,v)\in\mathcal{P}_{a_1,a_2}$, there are $(u^*,v^*)\in S_{a_1,r}\times S_{a_2,r}$ and $P(u^*,v^*)\leq P(u,v)=0$, which implies $t_* = t_{(u^*,v^*)}\leq 0$. It follows that

$$I_\mu(u,v) \geq I_\mu(t_*\star(u,v)) \geq I_\mu(t_*\star(u^*,v^*)) = \max_{s\in\mathbb{R}} I_\mu(s\star(u^*,v^*)).$$

Clearly,

$$\|\nabla s\star u^*\|_2^2 + \|\nabla s\star v^*\|_2^2 \to 0^+ \quad \text{as } s\to-\infty,$$

$$I_\mu\left(s\star\left(u^*,v^*\right)\right)\to-\infty\quad\text{as } s\to\infty.$$

So, there exist $s_0\ll -1$ and $s_1\gg 1$, such that $s_0\star(u^*,v^*)\in A_k$ and $s_1\star(u^*,v^*)\in I_\mu^0$. We define

$$\gamma_*: t\in[0,1]\mapsto\left(0,\left[(1-t)s_0+ts_1\right]\star(u^*,v^*)\right)\in\mathbb{R}\times S_{a_1,r}\times S_{a_2,r}.$$

By Lemma 4.3, one has $\gamma_*\in\Gamma$. Thus,

$$\sigma\le\max_{t\in[0,1]}\widetilde{I}_\mu(\gamma_*(t))\le\max_{s\in\mathbb{R}}I_\mu(s\star(u^*,v^*))\le I_\mu(u,v),$$

implying that $\sigma\le m_\mu(a_1,a_2)$. On the other hand, for any path $\gamma=(\alpha,\varphi_1,\varphi_2)\in\Gamma$, we consider the function

$$P_\gamma: t\in[0,1]\mapsto P\left(\alpha(t)\star(\varphi_1(t),\varphi_2(t))\right)\in\mathbb{R}.$$

One easily verify that $P_\gamma(0)>0$ and P_γ is continuous. We claim that $P_\gamma(1)<0$. Indeed, from Lemma 4.1, if $P_\gamma(1)\ge 0$, we get $t_{(\varphi_1(1),\varphi_2(1))}\ge 0$, and then

$$I_\mu(\varphi_1(1),\varphi_2(1))=\Phi_{(\varphi_1(1),\varphi_2(1))}(0)>\Phi_{(\varphi_1(1),\varphi_2(1))}(-\infty)=0^+,$$

which is a contradiction. Hence, we deduce that there exists $t_\gamma\in(0,1)$ such that $P_\gamma(t_\gamma)=0$, namely that, $\alpha(t_\gamma)\star(\varphi_1(t_\gamma),\varphi_2(t_\gamma))\in\mathcal{P}_{a_1,a_2}$, and so

$$\max_{t\in[0,1]}\tilde{I}_\mu(\gamma(t))\ge\tilde{I}_\mu\left(\gamma\left(t_\gamma\right)\right)=I_\mu\left(\alpha\left(t_\gamma\right)\star\left(\varphi_1\left(t_\gamma\right),\varphi_2\left(t_\gamma\right)\right)\right)\ge m_\mu\left(a_1,a_2\right),$$

which implies that $\sigma\ge m_\mu(a_1,a_2)$. Therefore, $\sigma=m_\mu(a_1,a_2)$.

Let $\mathcal{F}=\{\gamma([0,1]):\gamma\in\Gamma\}$. According to the notation of Theorem 3.2 by Ghoussoub [47], this means that $\mathcal{F}$ is a homotopy stable family of compact subsets of $\mathbb{R}\times S_{a_1,r}\times S_{a_2,r}$ with extended closed boundary $(\{0\}\times\overline{A}_k)\cup(\{0\}\times I_\mu^0)$, and that the superlevel set $\{\tilde{I}_\mu\ge\sigma\}$ is a dual set for $\mathcal{F}$, which means that the assumptions by Ghoussoub [47, Theorem 3.2] are satisfied. Therefore, by using [47, Theorem 3.2], we can take any minimizing sequence $\{\gamma_n([0,1]),\gamma_n=(\alpha_n,\varphi_{1,n},\varphi_{2,n})\}_{n\in\mathbb{N}}$ for σ with the property that $\alpha(t)=0$, $\varphi_{1,n}(t)\ge 0$, $\varphi_{2,n}(t)\ge 0$ for every $t\in[0,1]$. Indeed, replacing γ_n by $\tilde{\gamma}_n=(0,\alpha_n\star(|\varphi_{1,n}|,|\varphi_{2,n}|))$, there exists a Palais–Smale sequence $\{(s_n,u_n,v_n)\}_{n\in\mathbb{N}}\subseteq\mathbb{R}\times S_{a_1,r}\times S_{a_2,r}$, such that $\tilde{I}_\mu(s_n,u_n,v_n)\to\sigma$,

$$\partial_s\tilde{I}_\mu\left(s_n,u_n,v_n\right)\to 0\quad\text{and}\quad\left\|\partial_{(u,v)}\tilde{I}_\mu\left(s_n,u_n,v_n\right)\right\|_{(T_{u_n}S_{a_1,r}\times T_{v_n}S_{a_2,r})^*}\to 0,\tag{4.1}$$

as $n\to+\infty$, with the property that

$$\left|s_n\right|+\operatorname{dist}\left(\left(u_n,v_n\right),\left(\varphi_{1,n}([0,1]),\varphi_{2,n}([0,1])\right)\right)\to 0.\tag{4.2}$$

Let $(\overline{u}_n,\overline{v}_n)=s_n\star(u_n,v_n)\in S_{a_1,r}\times S_{a_2,r}$. By the definition of $\widetilde{I}_\mu(s_n,u_n,v_n)$ and the first condition in (4.1), we obtain $P(\overline{u}_n,\overline{v}_n)\to 0$. The second condition in (4.1) reveals that for any $(\phi,\psi)\in T_{\overline{u}_n}S_{a_1,r}\times T_{\overline{v}_n}S_{a_2,r}$, we have

$$\begin{aligned}I_\mu'\left(\overline{u}_n,\overline{v}_n\right)[\phi,\psi]&=\partial_{(u,v)}\tilde{I}_\mu\left(s_n,u_n,v_n\right)\left[\left(-s_n\right)\star(\phi,\psi)\right]\\&=o_n(1)\left\|\left(-s_n\right)\star(\phi,\psi)\right\|_H\\&=o_n(1)\|(\phi,\psi)\|_H\quad\text{as } n\to+\infty.\end{aligned}$$

From (4.2), we infer that $\{s_n\}_{n\in\mathbb{N}}$ is bounded and $\overline{u}_n^-,\overline{v}_n^-\to 0$ a.e. in $\mathbb{R}^N$. To sum up, $\{(\overline{u}_n,\overline{v}_n)\}_{n\in\mathbb{N}}$ is a Palais–Smale sequence for $I_\mu|_{S_{a_1,r}\times S_{a_2,r}}$ and hence a radial symmetric Palais–Smale sequence for $I_\mu|_{S_{a_1}\times S_{a_2}}$ at level $\sigma=m_\mu(a_1,a_2)$ with $P(\overline{u}_n,\overline{v}_n)\to 0$. □

We fix $a_1,a_2>0$. Then, using Lemma 2.1 by Li-Zou [43, Lemma 2.1.], if $q\in(2+\frac{4}{N},2^*)$, we have

$$m_\mu(a_1,0)>0\quad\text{and}\quad m_\mu(0,a_2)>0.\tag{4.3}$$

Lemma 4.5. *For fixed $a_1,a_2>0$, the following statements hold:*

(i) $m_\mu(a_1,a_2)\le m_\mu\left(b_1,b_2\right)$ *for any* $0<b_1\le a_1, 0<b_2\le a_2$*;*
(ii) $m_\mu(a_1,a_2)$ *is nonincreasing with respect to* $\mu\in(0,+\infty)$*;*
(iii) $\lim_{\mu\to\infty}m_\mu(a_1,a_2)=0^+$.

Proof. (i) The statements can be shown by Lemma 3.4 (ii). (ii) For any $\mu\ge\mu'>0$, one has

$$m_\mu(a_1,a_2)=\inf_{S_{a_1}\times S_{a_2}}\max_{s\in\mathbb{R}}I_\mu(s\star(u,v))\le\inf_{S_{a_1}\times S_{a_2}}\max_{s\in\mathbb{R}}I_{\mu'}(s\star(u,v))=m_{\mu'}(a_1,a_2),$$

from which the conclusion follows. (iii) We first prove that $m_\mu(a_1,a_2)>0$ for any $\mu>0$. Indeed, for any $(u,v)\in\mathcal{P}_{a_1,a_2}$, we have

$$\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)\,\mathrm{d}x=\mu\gamma_q\int_{\mathbb{R}^N}\left(|u|^q+|v|^q\right)\,\mathrm{d}x+2\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x$$

$$\leq C_1 \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2 \right) \mathrm{d}x \right)^{\frac{q\gamma_q}{2}} + C_2 \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2 \right) \mathrm{d}x \right)^{\frac{2^*}{2}},$$

which implies

$$\inf_{\mathcal{P}_{a_1,a_2}} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2 \right) \mathrm{d}x > 0.$$

Therefore, we deduce from this that

$$\begin{aligned} m_\mu(a_1,a_2) &= \inf_{\mathcal{P}_{a_1,a_2}} I_\mu(u,v) - \frac{1}{2} P(u,v) \\ &= \inf_{\mathcal{P}_{a_1,a_2}} \left(\mu \frac{q\gamma_q - 2}{2q} \int_{\mathbb{R}^N} \left(|u|^q + |v|^q \right) \mathrm{d}x + (1 - \frac{2}{2^*}) \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \,\mathrm{d}x \right) \\ &\geq C \inf_{\mathcal{P}_{a_1,a_2}} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2 \right) \mathrm{d}x > 0. \end{aligned}$$

Now (iii) holds if we can prove that for any $\varepsilon > 0$, there exists $\overline{\mu} > 0$ such that

$$m_\mu(a_1,a_2) < \varepsilon \quad \text{for any } \mu \geq \overline{\mu}. \tag{4.4}$$

Choosing $\phi \in C_0^\infty(\mathbb{R}^N)$ with $\|\phi\|_2 \leq \min\{a_1,a_2\}$ and noting that

$$\begin{aligned} m_\mu(a_1,a_2) &\leq m_\mu \left(\|\phi\|_2, \|\phi\|_2 \right) \leq \max_{s\in\mathbb{R}} I_\mu(s \star \phi, s \star \phi) \\ &= \max_{s\in\mathbb{R}} 2 \left(E(s\star\phi) - \frac{\mu e^{q\gamma_q s}}{q} \|\phi\|_q^q \right), \end{aligned} \tag{4.5}$$

where $E(u)$ is defined by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} \,\mathrm{d}x.$$

Since $E(s\star\phi) - \frac{\mu e^{q\gamma_q s}}{q} \|\phi\|_q^q \to 0^+$ as $s \to -\infty$, there exists $s_0 > 0$ such that $E(s\star\phi) - \frac{\mu e^{q\gamma_q s}}{q} \|\phi\|_q^q < \varepsilon$ for any $s < -s_0$. On the other hand, there exists $\overline{\mu} > 0$ such that

$$\begin{aligned} &\max_{s \geq -s_0} 2 \left(E(s\star\phi) - \frac{\mu e^{q\gamma_q s}}{q} \|\phi\|_q^q \right) \\ &\leq 2 \left(\frac{\|\nabla\phi\|_2^N}{N \|\phi\|_{2^*}^N} - \frac{\mu e^{-q\gamma_q s_0}}{q} \|\phi\|_q^q \right) < \varepsilon \quad \text{for } \mu \geq \overline{\mu}. \end{aligned}$$

and so $\max_{s\in\mathbb{R}} 2 \left(E(s\star\phi) - \frac{\mu e^{q\gamma_q s}}{q} \|\phi\|_q^q \right) < \varepsilon$ when $\mu \geq \overline{\mu}$. Therefore, by combining this with (4.5), we obtain (4.4), and the conclusion follows. □

Recall that the minimizer for S in (2.3) is achieved by the function

$$U_\varepsilon(x) := (N(N-2))^{\frac{N-2}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{\frac{N-2}{2}},$$

where $\varepsilon > 0$ is a parameter. We define the test functions $\eta_\varepsilon(x) := \phi(x) U_\varepsilon(x)$, where $\phi(x) \in C_0^\infty(\mathbb{R}^N)$ is a radial cut-off function with $\phi \in [0,1]$, $\phi \equiv 1$ on $B_1(0)$ and $\phi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$. By Jeanjean-Le [30] or Soave [26], we can derive the following well-known asymptotic estimations.

Lemma 4.6. *We have for $\varepsilon \to 0^+$*

$$\begin{aligned} \|\nabla \eta_\varepsilon\|_2^2 &= S^{\frac{N}{2}} + O(\varepsilon^{N-2}), \\ \|\eta_\varepsilon\|_{2^*}^{2^*} &= S^{\frac{N}{2}} + O(\varepsilon^N), \\ \|\eta_\varepsilon\|_2^2 &= \begin{cases} O(\varepsilon^2), & \text{if } N \geq 5, \\ O(\varepsilon^2 |\ln\varepsilon|), & \text{if } N = 4, \\ O(\varepsilon), & \text{if } N = 3, \end{cases} \end{aligned}$$

and

$$\|\eta_\varepsilon\|_q^q = \begin{cases} O(\varepsilon^{N-(N-2)q/2}), & \text{if } N \geq 4 \text{ and } q \in (2,2^*) \text{ or if } N = 3 \text{ and } q \in (3,6), \\ O(\varepsilon^{q/2}), & \text{if } N = 3 \text{ and } q \in (2,3), \\ O(\varepsilon^{3/2} |\ln\varepsilon|), & \text{if } N = 3 \text{ and } q = 3. \end{cases}$$

Lemma 4.7. *Fix* $a_1, a_2 > 0$, *then*

$$m_\mu(a_1,a_2) < \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}} \text{ for all } \mu > 0.$$

Proof. We define

$$u_\varepsilon = \frac{a_1}{\|\eta_\varepsilon\|_2}\eta_\varepsilon \quad \text{and} \quad v_\varepsilon = \frac{a_2}{\|\eta_\varepsilon\|_2}\eta_\varepsilon,$$

Let $t_\varepsilon := t_{(u_\varepsilon,v_\varepsilon)}$ be given by Lemma 4.1. Then by (2.6) and $t_\varepsilon \star (u_\varepsilon, v_\varepsilon) \in \mathcal{P}_{a_1,a_2}$, for $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned}
&m_\mu(a_1,a_2)\\
&\le I_\mu(t_\varepsilon \star (u_\varepsilon,v_\varepsilon))\\
&= \frac{e^{2t_\varepsilon}}{2}\int_{\mathbb{R}^N}\left(|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\right)\mathrm{d}x - \frac{\mu e^{q\gamma_q t_\varepsilon}}{q}\int_{\mathbb{R}^N}\left(|u_\varepsilon|^q+|v_\varepsilon|^q\right)\mathrm{d}x\\
&\quad - \frac{2e^{2^* t_\varepsilon}}{2^*}\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\,\mathrm{d}x\\
&\le \sup_{s>0}\left(\frac{s^2}{2}\int_{\mathbb{R}^N}\left(|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\right)\mathrm{d}x - \frac{2s^{2^*}}{2^*}\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\,\mathrm{d}x\right)\\
&\quad - Ce^{q\gamma_q t_\varepsilon}\frac{\|\eta_\varepsilon\|_q^q}{\|\eta_\varepsilon\|_2^q}\\
&= \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}} + O(\varepsilon^{N-2}) - Ce^{q\gamma_q t_\varepsilon}\frac{\|\eta_\varepsilon\|_q^q}{\|\eta_\varepsilon\|_2^q},
\end{aligned} \tag{4.6}$$

where we have used that

$$\sup_{s>0}\left(\frac{s^2}{2}A - \frac{2s^{2^*}}{2^*}B\right) = \frac{1}{N}\left(\frac{A}{(2B)^{2/2^*}}\right)^{N/2} \quad \text{with } A, B > 0.$$

According to $P(t_\varepsilon \star (u_\varepsilon, v_\varepsilon)) = 0$, we have that

$$2e^{(2^*-2)t_\varepsilon}\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\,\mathrm{d}x \le \int_{\mathbb{R}^N}\left(|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\right)\mathrm{d}x.$$

Hence it follows that

$$e^{t_\varepsilon} \le \left(\frac{\int_{\mathbb{R}^N}\left(|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\right)\mathrm{d}x}{2\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\,\mathrm{d}x}\right)^{\frac{1}{2^*-2}}. \tag{4.7}$$

From (4.7), $P(t_\varepsilon \star (u_\varepsilon, v_\varepsilon)) = 0$ and the fact $q\gamma_q > 2$, we infer that

$$\begin{aligned}
&e^{(2^*-2)t_\varepsilon}\\
&= \frac{\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\,\mathrm{d}x}{2\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\,\mathrm{d}x} - \mu\gamma_q e^{(q\gamma_q-2)t_\varepsilon}\frac{\int_{\mathbb{R}^N}|u_\varepsilon|^q+|v_\varepsilon|^q\,\mathrm{d}x}{2\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\,\mathrm{d}x}\\
&\ge \frac{\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\,\mathrm{d}x}{2\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\,\mathrm{d}x}\\
&\quad - \mu\gamma_q\frac{\int_{\mathbb{R}^N}|u_\varepsilon|^q+|v_\varepsilon|^q\,\mathrm{d}x}{2\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\,\mathrm{d}x}\left(\frac{\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\,\mathrm{d}x}{2\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta\,\mathrm{d}x}\right)^{\frac{q\gamma_q-2}{2^*-2}}\\
&= \frac{\|\nabla\eta_\varepsilon\|_2^2\|\eta_\varepsilon\|_2^{2^*-2}}{\|\eta_\varepsilon\|_{2^*}^{2^*}}\left(C_1 - C_2\|\nabla\eta_\varepsilon\|_2^{-\frac{2(2^*-q\gamma_q)}{2^*-2}}\|\eta_\varepsilon\|_{2^*}^{-\frac{2^*(q\gamma_q-2)}{2^*-2}}\frac{\|\eta_\varepsilon\|_q^q}{\|\eta_\varepsilon\|_2^{q-q\gamma_q}}\right).
\end{aligned}$$

On account of Lemma 4.6, by using $\|\nabla\eta_\varepsilon\|_2^2 \to S^{\frac{N}{2}}$, $\|\eta_\varepsilon\|_{2^*}^{2^*} \to S^{\frac{N}{2}}$ and

$$\frac{\|\eta_\varepsilon\|_q^q}{\|\eta_\varepsilon\|_2^{q-q\gamma_q}} = \begin{cases} O(\varepsilon^{\frac{6-q}{4}}), & \text{if } N = 3,\\ O(|\ln\varepsilon|^{-\frac{q(1-\gamma_q)}{2}}), & \text{if } N = 4,\end{cases}$$

as $\varepsilon \to 0$, for some constant $C > 0$, we can obtain

$$e^{t_\varepsilon} \ge C\|\eta_\varepsilon\|_2.$$

Consequently, by using (4.6), for all sufficiently small $\varepsilon > 0$, we infer to

$$\begin{aligned} m_\mu(a_1,a_2) &\le \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}} + O(\varepsilon^{N-2}) - C\frac{\|\eta_\varepsilon\|_q^q}{\|\eta_\varepsilon\|_2^{q-q\gamma_q}} \\ &\le \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}} + O(\varepsilon^{N-2}) - \begin{cases} O(\varepsilon^{\frac{6-q}{4}}), & \text{if } N=3, \\ O(|\ln\varepsilon|^{-\frac{q(1-\gamma_q)}{2}}), & \text{if } N=4, \end{cases} \\ &< \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}, \end{aligned}$$

from which the statement of the lemma follows. □

Proof of Theorem 1.2. Combining (4.3) along with Lemmas 2.5, 4.4, 4.5 and 4.7, the proof is complete. □

5. Proof of Theorem 1.3

In this section, we deal with the L^2-critical case $q = \bar{q} := 2 + \frac{4}{N} < 2^*$. Note that $\bar{q}\gamma_{\bar{q}} = 2$. We first introduce the following constant

$$\mu_3 := \frac{\bar{q}}{2} C_{N,\bar{q}}^{-1}\left(a_1^{\bar{q}-2} + a_2^{\bar{q}-2}\right)^{-1}. \tag{5.1}$$

Lemma 5.1. *For $\mu > 0$, we have*

$$\mathcal{O}_\mu(a_1,a_2) := \left\{(u,v) \in S_{a_1}\times S_{a_2} : \|\nabla u\|_2^2 + \|\nabla v\|_2^2 > \frac{2\mu}{\bar{q}}(\|u\|_{\bar{q}}^{\bar{q}} + \|v\|_{\bar{q}}^{\bar{q}})\right\} \ne \emptyset.$$

If $0 < \mu < \mu_3$, then $\mathcal{O}_\mu(a_1,a_2) = S_{a_1}\times S_{a_2}$.

Proof. If $0 < \mu < \mu_3$, for $(u,v) \in S_{a_1}\times S_{a_2}$, we get

$$\frac{2\mu}{\bar{q}}(\|u\|_{\bar{q}}^{\bar{q}} + \|v\|_{\bar{q}}^{\bar{q}}) \le \frac{2\mu}{\bar{q}} C_{N,\bar{q}}(a_1^{\bar{q}-2} + a_2^{\bar{q}-2})(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) < (\|\nabla u\|_2^2 + \|\nabla v\|_2^2),$$

hence $S_{a_1}\times S_{a_2} = \mathcal{O}_\mu(a_1,a_2)$. If $\mu \ge \mu_3$, we claim that

$$\sup_{(u,v)\in S_{a_1}\times S_{a_2}} \frac{\|\nabla u\|_2^2 + \|\nabla v\|_2^2}{\|u\|_{\bar{q}}^{\bar{q}} + \|v\|_{\bar{q}}^{\bar{q}}} = +\infty, \tag{5.2}$$

from which $\mathcal{O}_\mu(a_1,a_2) \ne \emptyset$ follows. For this purpose, we consider the function

$$\phi_n(x) := \begin{cases} \sin n|x|, & \text{if } |x| \le \pi, \\ 0, & \text{if } |x| > \pi, \end{cases}$$

and define $u_n := \frac{a_1}{\|\phi_n\|_2}\phi_n$, $v_n := \frac{a_2}{\|\phi_n\|_2}\phi_n$. Clearly, $(u_n,v_n) \in S_{a_1}\times S_{a_2}$. Since $\|\nabla\phi_n\|_2^2 = O(n^2)$, $\|\phi_n\|_2^2 = O(1)$ and $\|\phi_n\|_{\bar{q}}^{\bar{q}} = O(1)$, one has

$$\frac{\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2}{\|u_n\|_{\bar{q}}^{\bar{q}} + \|v_n\|_{\bar{q}}^{\bar{q}}} = \frac{a_1^2 + a_2^2}{a_1^{\bar{q}} + a_2^{\bar{q}}} \frac{\|\phi_n\|_2^{\bar{q}-2}\|\nabla\phi_n\|_2^2}{\|\phi_n\|_{\bar{q}}^{\bar{q}}} \to +\infty \quad \text{as } n \to +\infty,$$

and (5.2) follows. □

Recall the decomposition

$$\mathcal{P}_{a_1,a_2} = \mathcal{P}^+_{a_1,a_2} \cup \mathcal{P}^0_{a_1,a_2} \cup \mathcal{P}^-_{a_1,a_2},$$

introduced in (2.9). Then, we have the following conclusions, by analogous arguments as in Section 4.

Lemma 5.2. *For each $(u,v) \in \mathcal{O}_\mu(a_1,a_2)$, there exists a unique $t_{(u,v)} \in \mathbb{R}$ such that $t_{(u,v)} \star (u,v) \in \mathcal{P}_{a_1,a_2}$, where $t_{(u,v)}$ is the unique critical point of the function of $\Phi_{(u,v)}$. Moreover, it holds:*

(i) *$\mathcal{P}_{a_1,a_2} = \mathcal{P}^-_{a_1,a_2}$ and $\mathcal{P}_{a_1,a_2}$ is a submanifold of H;*
(ii) *$s \star (u,v) \in \mathcal{P}_{a_1,a_2}$ if and only if $s = t_{(u,v)}$;*
(iii) *$\Phi_{(u,v)}(s)$ is strictly decreasing on $(t_{(u,v)}, +\infty)$ and*

$$\Phi_{(u,v)}(t_{(u,v)}) = \max_{s\in\mathbb{R}} \Phi_{(u,v)}(s) > 0;$$

(iv) *The map* $(u, v) \mapsto t_{(u,v)} \in \mathbb{R}$ *is of class* C^1.

Lemma 5.3. $m_\mu(a_1, a_2)$ *has a minimax representation of the form*

$$m_\mu(a_1, a_2) = \inf_{\mathcal{O}_\mu(a_1,a_2)} \max_{s\in\mathbb{R}} I_\mu(s \star (u, v)).$$

From Soave [25], we know that for fixed $\lambda, \mu > 0$,

$$m_\mu(a_1, 0) > 0 \quad \text{and} \quad m_\mu(0, a_2) > 0. \tag{5.3}$$

Lemma 5.4. *If* $0 < \mu < \mu_3$*, then there exists a radial Palais–Smale sequence for* $I_\mu|_{\mathcal{O}_\mu(a_1,a_2)}$ *at level* $m_\mu(a_1, a_2)$ *with* $P\left(u_n, v_n\right) \to 0$ *and* $u_n^-, v_n^- \to 0$ *a.e. in* $\mathbb{R}^N$.

The proofs of Lemmas 5.2, 5.3 and 5.4 can be carried out analogously as it was done in the proofs of Lemmas 4.1, 4.2 and 4.4, respectively. So we omit it here.

Lemma 5.5. *Let* $0 < \mu < \mu_3$ *and fix* $a_1, a_2 > 0$*. Then the following statements hold:*

(i) $m_\mu(a_1, a_2) \le m_\mu\left(b_1, b_2\right)$ *for any* $0 < b_1 \le a_1, 0 < b_2 \le a_2$*;*
(ii) $m_\mu(a_1, a_2)$ *is nonincreasing with respect to* $\mu \in (0, \mu_3)$*.*

Proof. (i) Since $0 < \mu < \mu_3$ implies $\mathcal{O}_\mu(a_1, a_2) = S_{a_1} \times S_{a_2}$, the statement can be proved as in Lemma 3.4 (ii).
(ii) For any $\mu_3 > \mu \ge \mu' > 0$, one has

$$m_\mu(a_1, a_2) = \inf_{S_{a_1}\times S_{a_2}} \max_{s\in\mathbb{R}} I_\mu(s \star (u, v)) \le \inf_{S_{a_1}\times S_{a_2}} \max_{s\in\mathbb{R}} I_{\mu'}(s \star (u, v)) = m_{\mu'}(a_1, a_2),$$

from which the assertion follows. □

As in the previous section, the following estimate will play a crucial role in the proof of the existence of a ground state.

Lemma 5.6. *Let* $0 < \mu < \mu_3$ *and fix* $a_1, a_2 > 0$*. Then it holds*

$$m_\mu(a_1, a_2) < \frac{2}{N} \left(\frac{S_{\alpha,\beta}}{2} \right)^{\frac{N}{2}}.$$

Proof. We first prove that $m_\mu(a_1, a_2) > 0$ for $0 < \mu < \mu_3$. Indeed, for any $(u, v) \in \mathcal{P}_{a_1,a_2}$, by (2.7) and (5.1), we have

$$\begin{aligned}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) \mathrm{d}x &= \frac{2\mu}{\bar{q}} \int_{\mathbb{R}^N} \left(|u|^{\bar{q}} + |v|^{\bar{q}}\right) \mathrm{d}x + 2 \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \,\mathrm{d}x \\ &\le \frac{\mu}{\mu_3} \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) \mathrm{d}x \right) \\ &\quad + C \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) \mathrm{d}x \right)^{\frac{2^*}{2}},\end{aligned}$$

which implies

$$\inf_{(u,v)\in\mathcal{P}_{a_1,a_2}} \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \,\mathrm{d}x > 0.$$

Therefore, we deduce that

$$m_\mu(a_1, a_2) = \inf_{\mathcal{P}_{a_1,a_2}} I_\mu(u, v) - \frac{1}{2} P(u, v) = \frac{2}{N} \inf_{\mathcal{P}_{a_1,a_2}} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \,\mathrm{d}x > 0.$$

Recall the test functions $\eta_\varepsilon(x) := \phi(x) U_\varepsilon(x)$ and the definitions

$$u_\varepsilon = \frac{a_1}{\|\eta_\varepsilon\|_2} \eta_\varepsilon \quad \text{and} \quad v_\varepsilon = \frac{a_2}{\|\eta_\varepsilon\|_2} \eta_\varepsilon$$

from Section 4. Let $t_\varepsilon := t_{(u_\varepsilon, v_\varepsilon)}$ be given by Lemma 5.2. Then by (2.6) and $t_\varepsilon \star (u_\varepsilon, v_\varepsilon) \in \mathcal{P}_{a_1,a_2}$, for $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned} m_\mu(a_1,a_2) &\le I_\mu(t_\varepsilon \star (u_\varepsilon, v_\varepsilon)) \\ &= \frac{e^{2t_\varepsilon}}{2}\int_{\mathbb{R}^N}\left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right) dx - \frac{\mu e^{2t_\varepsilon}}{\bar q}\int_{\mathbb{R}^N}\left(|u_\varepsilon|^{\bar q} + |v_\varepsilon|^{\bar q}\right) dx \\ &\quad - \frac{2e^{2^* t_\varepsilon}}{2^*}\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta dx \\ &\le \sup_{s>0}\left(\frac{s^2}{2}\int_{\mathbb{R}^N}\left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right) dx - \frac{2s^{2^*}}{2^*}\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta dx\right) \\ &\quad - Ce^{2t_\varepsilon}\frac{\|\eta_\varepsilon\|_{\bar q}^{\bar q}}{\|\eta_\varepsilon\|_2^{\bar q}} \\ &= \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}} + O(\varepsilon^{N-2}) - Ce^{2t_\varepsilon}\frac{\|\eta_\varepsilon\|_{\bar q}^{\bar q}}{\|\eta_\varepsilon\|_2^{\bar q}}, \end{aligned} \tag{5.4}$$

where we have used again the fact that

$$\sup_{s>0}\left(\frac{s^2}{2}A - \frac{2s^{2^*}}{2^*}B\right) = \frac{1}{N}\left(\frac{A}{(2B)^{2/2^*}}\right)^{N/2} \quad \text{with } A, B > 0.$$

According to $P(t_\varepsilon \star (u_\varepsilon, v_\varepsilon)) = 0$ and the fact $\bar q \gamma_{\bar q} = 2$, for some constant $C > 0$, we infer that

$$\begin{aligned} e^{(2^*-2)t_\varepsilon} &= \frac{\int_{\mathbb{R}^N}\left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right) dx}{2\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta dx} - \mu\gamma_{\bar q}\frac{\int_{\mathbb{R}^N}\left(|u_\varepsilon|^{\bar q} + |v_\varepsilon|^{\bar q}\right) dx}{2\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta dx} \\ &\ge \left(1 - \mu\gamma_{\bar q}C_{N,\bar q}(a_1^{\bar q-2} + a_2^{\bar q-2})\right)\frac{\int_{\mathbb{R}^N}\left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right) dx}{2\int_{\mathbb{R}^N}|u_\varepsilon|^\alpha|v_\varepsilon|^\beta dx} \\ &= C\frac{\|\nabla\eta_\varepsilon\|_2^2\|\eta_\varepsilon\|_2^{2^*-2}}{\|\eta_\varepsilon\|_{2^*}^{2^*}}, \end{aligned}$$

where we used $0 < \mu < \mu_3$. Furthermore, on account of Lemma 4.6, by using $\|\nabla\eta_\varepsilon\|_2^2 \to S^{\frac{N}{2}}$ and $\|\eta_\varepsilon\|_{2^*}^{2^*} \to S^{\frac{N}{2}}$ as $\varepsilon \to 0$, for some constant $C > 0$, we have

$$e^{t_\varepsilon} \ge C\|\eta_\varepsilon\|_2. \tag{5.5}$$

Thus, in conjunction with (5.4) and (5.5), for $\varepsilon > 0$ sufficiently small, yields

$$\begin{aligned} m_\mu(a_1,a_2) &\le \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}} + O(\varepsilon^{N-2}) - C\frac{\|\eta_\varepsilon\|_{\bar q}^{\bar q}}{\|\eta_\varepsilon\|_2^{\bar q-2}} \\ &\le \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}} + O(\varepsilon^{N-2}) - \begin{cases} O(\varepsilon^{\frac{6-\bar q}{4}}), & \text{if } N = 3, \\ O(|\ln\varepsilon|^{-\frac{\bar q(1-\gamma_{\bar q})}{2}}), & \text{if } N = 4, \end{cases} \\ &\le \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}} + O(\varepsilon^{N-2}) - \begin{cases} O(\varepsilon^{\frac{2}{3}}), & \text{if } N = 3, \\ O(|\ln\varepsilon|^{-\frac{1}{2}}), & \text{if } N = 4, \end{cases} \\ &< \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}. \end{aligned}$$

This proves the assertion of the lemma. □

Proof of Theorem 1.3. The proof follows by combining (5.3) with Lemmas 2.5, 5.4, 5.5 and 5.6. □

6. Proof of Theorem 1.4

In this section, we deal with the L^2-supercritical case $2 + \frac{4}{N} < q < 2^*$ when $\mu > 0$ is large enough. Recalling the strategy introduced by Jeanjean [24] and consider the auxiliary functional

$$\widetilde{I}_\mu : \mathbb{R} \times S_{a_1} \times S_{a_2} \to \mathbb{R}, \quad (s, u, v) \mapsto I_\mu(s \star (u, v)),$$

where

$$s \star (u, v)(x) = (s \star u, s \star v)(x) = \left(e^{\frac{Ns}{2}}u(e^s x), e^{\frac{Ns}{2}}v(e^s x)\right).$$

By a direct calculation, we have

$$\int_{\mathbb{R}^N} |\nabla(s\star(u,v))|^2\,\mathrm{d}x = e^{2s}\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)\mathrm{d}x,$$
$$\int_{\mathbb{R}^N} |s\star(u,v)|^q\,\mathrm{d}x = e^{\frac{N(q-2)}{2}s}\int_{\mathbb{R}^N}\left(|u|^q+|v|^q\right)\mathrm{d}x \quad \text{for all } q\in[2,2^*],$$

and

$$\int_{\mathbb{R}^N} |s\star u|^\alpha|s\star v|^\beta\,\mathrm{d}x = e^{2^*s}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x.$$

Then, it follows that

$$\begin{aligned}\widetilde{I}_\mu(s,u,v) &= I_\mu(s\star(u,v)) = I_\mu\left(e^{\frac{Ns}{2}}u(e^s x), e^{\frac{Ns}{2}}v(e^s x)\right)\\ &= \frac{e^{2s}}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)\mathrm{d}x - \frac{\mu e^{q\gamma_q s}}{q}\int_{\mathbb{R}^N}\left(|u|^q+|v|^q\right)\mathrm{d}x\\ &\quad - \frac{2e^{2^*s}}{2^*}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x.\end{aligned}$$

Summarizing the above expressions, we have the following result.

Lemma 6.1. *For any fixed $(u,v)\in S_{a_1}\times S_{a_2}$, there hold*

(i) $\int_{\mathbb{R}^N}|\nabla(s\star(u,v))|^2\,\mathrm{d}x\to 0$ *and* $I_\mu(s\star(u,v))\to 0$ *as* $s\to-\infty$;

(ii) $\int_{\mathbb{R}^N}|\nabla(s\star(u,v))|^2\,\mathrm{d}x\to +\infty$ *and* $I_\mu(s\star(u,v))\to -\infty$ *as* $s\to+\infty$.

Lemma 6.2. *There exists $K(a_1,a_2)>0$ sufficiently small such that*

$$I_\mu(u,v)>0 \quad \textit{for } u\in K_1 \quad \textit{and} \quad 0<\sup_{(u,v)\in K_1} I_\mu(u,v) < \inf_{(u,v)\in K_2} I_\mu(u,v),$$

where

$$\begin{aligned} K_1 &= \left\{(u,v)\in S_{a_1}\times S_{a_2} : \int_{\mathbb{R}^N}|\nabla u|^2+|\nabla v|^2\,\mathrm{d}x \le K(a_1,a_2)\right\},\\ K_2 &= \left\{(u,v)\in S_{a_1}\times S_{a_2} : \int_{\mathbb{R}^N}|\nabla u|^2+|\nabla v|^2\,\mathrm{d}x = 2K(a_1,a_2)\right\}.\end{aligned} \tag{6.1}$$

Proof. Let $K>0$ be arbitrary but fixed and suppose $(u_1,v_1)\in S_{a_1}\times S_{a_2}$ and $(u_2,v_2)\in S_{a_1}\times S_{a_2}$ satisfying

$$\int_{\mathbb{R}^N}\left(|\nabla u_1|^2+|\nabla v_1|^2\right)\mathrm{d}x \le K \quad \text{and} \quad \int_{\mathbb{R}^N}\left(|\nabla u_2|^2+|\nabla v_2|^2\right)\mathrm{d}x = 2K.$$

Then, for sufficiently small $K>0$, by (2.1) and (2.5), it follows that

$$\begin{aligned} &I_\mu(u_2,v_2)-I_\mu(u_1,v_1)\\ &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_2|^2+|\nabla v_2|^2)\,\mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^N}(|u_2|^q+|v_2|^q)\,\mathrm{d}x - \frac{2}{2^*}\int_{\mathbb{R}^N}|u_2|^\alpha|v_2|^\beta\,\mathrm{d}x\\ &\quad - \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_1|^2+|\nabla v_1|^2)\,\mathrm{d}x + \frac{\mu}{q}\int_{\mathbb{R}^N}(|u_1|^q+|v_1|^q)\,\mathrm{d}x + \frac{2}{2^*}\int_{\mathbb{R}^N}|u_1|^\alpha|v_1|^\beta\,\mathrm{d}x\\ &\ge \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_2|^2+|\nabla v_2|^2)\,\mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^N}(|u_2|^q+|v_2|^q)\,\mathrm{d}x - \frac{2}{2^*}\int_{\mathbb{R}^N}|u_2|^\alpha|v_2|^\beta\,\mathrm{d}x\\ &\quad - \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_1|^2+|\nabla v_1|^2)\,\mathrm{d}x\\ &\ge \frac{K}{2} - C_1K^{\frac{2^*}{2}} - C_2K^{\frac{N(q-2)}{4}} \ge \frac{K}{4},\end{aligned}$$

where we used that $2^*>2$ and $\frac{N(q-2)}{4}>1$. On the other hand, for $K>0$ small enough, for any $(u_1,v_1)\in S_{a_1}\times S_{a_2}$ satisfying $\int_{\mathbb{R}^N}|\nabla u_1|^2+|\nabla v_1|^2\,\mathrm{d}x\le K$, from (2.1) and (2.5) again, we infer to

$$\begin{aligned} I_\mu(u_1,v_1) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u_1|^2+|\nabla v_1|^2\right)\mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^N}\left(|u_1|^q+|v_1|^q\right)\mathrm{d}x\\ &\quad - \frac{2}{2^*}\int_{\mathbb{R}^N}|u_1|^\alpha|v_1|^\beta\,\mathrm{d}x\\ &\ge \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u_1|^2+|\nabla v_1|^2\right)\mathrm{d}x - C_3\left(\int_{\mathbb{R}^N}\left(|\nabla u_1|^2+|\nabla v_1|^2\right)\mathrm{d}x\right)^{\frac{2^*}{2}}\end{aligned}$$

$$- C_4 \left(\int_{\mathbb{R}^N} \left(|\nabla u_1|^2 + |\nabla v_1|^2 \right) \mathrm{d}x \right)^{\frac{N(q-2)}{4}} .$$

Hence, we can choose a sufficiently small constant $K(a_1, a_2) > 0$ in (6.1) such that

$$I_\mu(u, v) > 0 \quad \text{for } u \in K_1 \quad \text{and} \quad 0 < \sup_{(u,v)\in K_1} I_\mu(u, v) < \inf_{(u,v)\in K_2} I_\mu(u, v),$$

where K_1 and K_2 are given by (6.1). □

On account of Lemmas 6.1 and 6.2, for fixed $(u_0, v_0) \in S_{a_1} \times S_{a_2}$, there exist two constants s_1, s_2 satisfying $s_1 \ll -1 < 0 < 1 \ll s_2$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla u_1|^2 + |\nabla v_1|^2 \right) \mathrm{d}x < \frac{K(a_1, a_2)}{2}, \quad \int_{\mathbb{R}^N} \left(|\nabla u_2|^2 + |\nabla v_2|^2 \right) \mathrm{d}x > 2K(a_1, a_2)$$

and

$$I_\mu(u_1, v_1) > 0, \quad I_\mu(u_2, v_2) < 0,$$

where $(u_1, v_1) := s_1 \star (u_0, v_0) \in S_{a_1} \times S_{a_2}$ and $(u_2, v_2) := s_2 \star (u_0, v_0) \in S_{a_1} \times S_{a_2}$. Based on this, we now define a minimax level

$$\gamma_\mu(a_1, a_2) := \inf_{g\in\Gamma} \max_{t\in[0,1]} I_\mu(g(t)),$$

where

$$\Gamma := \left\{ g \in C([0, 1], S_{a_1} \times S_{a_2}) : g(0) = (u_1, v_1), g(1) = (u_2, v_2) \right\}.$$

Then for any $g \in \Gamma$ there holds

$$\max_{t\in[0,1]} I_\mu(g(t)) > \max\{I_\mu(u_1, v_1), I_\mu(u_2, v_2)\},$$

which implies $\gamma_\mu(a_1, a_2) > 0$.

Lemma 6.3. *It holds* $\lim_{\mu\to+\infty} \gamma_\mu(a_1, a_2) = 0$.

Proof. For any fixed $(u_0, v_0) \in S_{a_1} \times S_{a_2}$, we set $g_0(t) := [(1 - t)s_1 + ts_2] \star (u_0, v_0) \in \Gamma$. Therefore, we can obtain

$$\begin{aligned}
0 < \gamma_\mu(a_1, a_2) &\le \max_{t\in[0,1]} I_\mu(g_0(t)) = \max_{t\in[0,1]} I_\mu \left([(1 - t)s_1 + ts_2] \star (u_0, v_0) \right) \\
&= \max_{t\in[0,1]} \left\{ \frac{1}{2} e^{2[(1-t)s_1+ts_2]} \int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + |\nabla v_0|^2 \right) \mathrm{d}x \right. \\
&\qquad - \frac{\mu}{q} e^{\frac{q-2}{2} N[(1-t)s_1+ts_2]} \int_{\mathbb{R}^N} \left(|u_0|^q + |v_0|^q \right) \mathrm{d}x \\
&\qquad \left. - \frac{2}{2^*} e^{2^*[(1-t)s_1+ts_2]} \int_{\mathbb{R}^N} |u_0|^\alpha |v_0|^\beta \, \mathrm{d}x \right\} \\
&\le \max_{r\ge0} \left\{ \frac{1}{2} r^2 \int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + |\nabla v_0|^2 \right) \mathrm{d}x - \frac{\mu}{q} r^{\frac{N(q-2)}{2}} \int_{\mathbb{R}^N} \left(|u_0|^q + |v_0|^q \right) \mathrm{d}x \right\} \\
&\le C \mu^{-\frac{4}{N(q-2)-4}} \to 0, \text{ as } \mu \to +\infty,
\end{aligned}$$

since $2 + \frac{4}{N} < q < 2^*$. □

For convenience, we set $f(t_1, t_2) = \mu|t_1|^{q-2}t_1 + \mu|t_2|^{q-2}t_2$ for any $t_1, t_2 \in \mathbb{R}$. Utilizing the same argument as Proposition 2.2 by Jeanjean [24], there exist a Palais–Smale sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subseteq S_{a_1} \times S_{a_2}$ associated with the level $\gamma_\mu(a_1, a_2)$ such that

$$I_\mu(u_n, v_n) \to \gamma_\mu(a_1, a_2), \quad \|I'_\mu|_{S_{a_1}\times S_{a_2}}(u_n, v_n)\| \to 0 \quad \text{and} \quad P(u_n, v_n) \to 0, \tag{6.2}$$

as $n \to \infty$, where

$$\begin{aligned}
P(u_n, v_n) = &\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) \mathrm{d}x - 2 \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \, \mathrm{d}x \\
&+ N \int_{\mathbb{R}^N} F(u_n, v_n) \, \mathrm{d}x - \frac{N}{2} \int_{\mathbb{R}^N} f(u_n, v_n)(u_n, v_n) \, \mathrm{d}x,
\end{aligned} \tag{6.3}$$

with $f(u_n, v_n)(u_n, v_n) := \mu(|u_n|^q + |v_n|^q)$. According to Proposition 5.12 by Willem [36], there exists a sequence $\{(\lambda_{1,n}, \lambda_{2,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}\times\mathbb{R}$ such that

$$I'_\mu \left(u_n, v_n\right) + \lambda_{1,n}(u_n, 0) + \lambda_{2,n}(0, v_n) \to 0 \quad \text{as } n \to \infty. \tag{6.4}$$

Lemma 6.4. *There exists a constant $C = C(N,q) > 0$ such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} F(u_n, v_n)\,\mathrm{d}x \le C\gamma_\mu(a_1,a_2),$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} f(u_n, v_n)(u_n,v_n)\,\mathrm{d}x \le C\gamma_\mu(a_1,a_2),$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + |\nabla v_n|^2\,\mathrm{d}x \le C\gamma_\mu(a_1,a_2).$$

Proof. From $I_\mu(u_n,v_n) \to \gamma_\mu(a_1,a_2)$ and $P(u_n,v_n) \to 0$ as $n \to \infty$, we have

$$\begin{aligned}
&N\gamma_\mu(a_1,a_2) + o_n(1)\\
&= NI_\mu(u_n,v_n) + P(u_n,v_n)\\
&= \frac{N+2}{2}\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right)\,\mathrm{d}x - \frac{2N+22^*}{2^*}\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta\,\mathrm{d}x\\
&\quad - \frac{N}{2}\int_{\mathbb{R}^N} f(u_n,v_n)(u_n,v_n)\,\mathrm{d}x\\
&\le (N+2)\int_{\mathbb{R}^N} \left(\frac{1}{2}\left(|\nabla u_n|^2 + |\nabla v_n|^2\right) - \frac{2}{2^*}|u_n|^\alpha |v_n|^\beta\right)\,\mathrm{d}x\\
&\quad - \frac{N}{2}\int_{\mathbb{R}^N} f(u_n,v_n)(u_n,v_n)\,\mathrm{d}x\\
&= (N+2)[\gamma_\mu(a_1,a_2) + \int_{\mathbb{R}^N} F(u_n,v_n)\,\mathrm{d}x + o_n(1)]\\
&\quad - \frac{N}{2}\int_{\mathbb{R}^N} f(u_n,v_n)(u_n,v_n)\,\mathrm{d}x,
\end{aligned}$$

which leads to

$$\begin{aligned}
2\gamma_\mu(a_1,a_2) + o_n(1) &\ge \frac{N}{2}\int_{\mathbb{R}^N} f(u_n,v_n)(u_n,v_n)\,\mathrm{d}x - (N+2)\int_{\mathbb{R}^N} F(u_n,v_n)\,\mathrm{d}x\\
&= \frac{Nq}{2}\int_{\mathbb{R}^N} F(u_n,v_n)\,\mathrm{d}x - (N+2)\int_{\mathbb{R}^N} F(u_n,v_n)\,\mathrm{d}x\\
&= \frac{Nq - 2(N+2)}{2}\int_{\mathbb{R}^N} F(u_n,v_n)\,\mathrm{d}x.
\end{aligned}$$

That is,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} F(u_n,v_n)\,\mathrm{d}x \le \frac{4}{Nq - 2(N+2)}\gamma_\mu(a_1,a_2), \tag{6.5}$$

and thus

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} f(u_n,v_n)(u_n,v_n)\,\mathrm{d}x \le C\gamma_\mu(a_1,a_2). \tag{6.6}$$

Again by (6.2) and (6.3), we infer that

$$\begin{aligned}
\gamma_\mu(a_1,a_2) + o_n(1) &= I_\mu(u_n,v_n) - \frac{1}{2^*}P(u_n,v_n)\\
&= \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right)\,\mathrm{d}x\\
&\quad - \left(1 + \frac{N}{2^*}\right)\int_{\mathbb{R}^N} F(u_n,v_n)\,\mathrm{d}x\\
&\quad + \frac{N}{22^*}\int_{\mathbb{R}^N} f(u_n,v_n)(u_n,v_n)\,\mathrm{d}x.
\end{aligned} \tag{6.7}$$

Consequently, by combining (6.5), (6.6) and (6.7), we immediately obtain

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right)\,\mathrm{d}x \le C\gamma_\mu(a_1,a_2). \qquad \square$$

In view of the boundedness of $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ in H_r and Lemma 2.1, up to a subsequence, there exists $(u,v) \in S_{a_1,r} \times S_{a_2,r}$ such that

$$\begin{cases}
(u_n,v_n) \rightharpoonup (u,v), & \text{in } H_r,\\
(u_n,v_n) \to (u,v), & \text{in } L^t\left(\mathbb{R}^N\right) \times L^t\left(\mathbb{R}^N\right) \text{ for } t \in (2,2^*),\\
(u_n,v_n) \to (u,v), & \text{a.e. in } \mathbb{R}^N,
\end{cases} \tag{6.8}$$

as $n \to \infty$. Using $2 + \frac{4}{N} < q < 2^*$, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^q\,\mathrm{d}x = \int_{\mathbb{R}^N} |u|^q\,\mathrm{d}x \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^q\,\mathrm{d}x = \int_{\mathbb{R}^N} |v|^q\,\mathrm{d}x.$$

Now, take $(u_n, 0)$ and $(0, v_n)$ as test functions in (6.4), we see that $\{(\lambda_{1,n}, \lambda_{2,n})\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}\times\mathbb{R}$. Thus, there exists $(\lambda_1, \lambda_2) \in \mathbb{R}\times\mathbb{R}$ such that, up to a subsequence, $(\lambda_{1,n}, \lambda_{2,n}) \to (\lambda_1, \lambda_2)$ as $n \to \infty$.

By using the concentration-compactness principle, we obtain the following result.

Lemma 6.5. *There exists $\mu^* > 0$ large enough such that for any $\mu > \mu^*$, one has*

$$\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \,\mathrm{d}x \to \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \,\mathrm{d}x \quad \text{as } n \to \infty.$$

Proof. By Lemmas 2.3 and 2.4, we divide the proof into three steps.

Step 1. We show that $\omega_j = 2\nu_j$, where ω_j and ν_j are given in Lemma 2.3.

Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with $\varphi \in [0,1]$, $\varphi \equiv 1$ in $B_{\frac{1}{2}}(0)$ and $\varphi \equiv 0$ in $\mathbb{R}^N \setminus B_1(0)$. For any $\rho > 0$, we set

$$\varphi_\rho(x) = \varphi\left(\frac{x - x_j}{\rho}\right) = \begin{cases} 1, & \text{if } |x - x_j| \le \frac{1}{2}\rho, \\ 0, & \text{if } |x - x_j| \ge \rho. \end{cases}$$

By Lemma 6.4, we know that $\{\varphi_\rho u_n\}_{n\in\mathbb{N}}$ and $\{\varphi_\rho v_n\}_{n\in\mathbb{N}}$ are bounded in $H^1(\mathbb{R}^N)$. Then, we choose $(\varphi_\rho u_n, \varphi_\rho v_n)$ as a test function in (6.4) and let $\rho \to 0$. This gives

$$\lim_{\rho\to 0}\lim_{n\to\infty} \langle I'_\mu\left(u_n, v_n\right) + \lambda_{1,n}(u_n, 0) + \lambda_{2,n}(0, v_n), (\varphi_\rho u_n, \varphi_\rho v_n)\rangle = 0. \tag{6.9}$$

From (6.8), the definition of φ_ρ and the absolute continuity of the Lebesgue integral, it follows that

$$\begin{aligned}\lim_{\rho\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} \varphi_\rho |u_n|^q \,\mathrm{d}x &= \lim_{\rho\to 0} \int_{\mathbb{R}^N} \varphi_\rho |u|^q \,\mathrm{d}x \\ &= \lim_{\rho\to 0} \int_{|x-x_j|\le\rho} \varphi_\rho |u|^q \,\mathrm{d}x = 0,\end{aligned} \tag{6.10}$$

$$\begin{aligned}\lim_{\rho\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} \varphi_\rho |v_n|^q \,\mathrm{d}x &= \lim_{\rho\to 0} \int_{\mathbb{R}^N} \varphi_\rho |v|^q \,\mathrm{d}x \\ &= \lim_{\rho\to 0} \int_{|x-x_j|\le\rho} \varphi_\rho |v|^q \,\mathrm{d}x = 0,\end{aligned} \tag{6.11}$$

$$\lim_{\rho\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} \lambda_{1,n}\varphi_\rho u_n^2 \,\mathrm{d}x = 0, \tag{6.12}$$

and

$$\lim_{\rho\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} \lambda_{2,n}\varphi_\rho v_n^2 \,\mathrm{d}x = 0. \tag{6.13}$$

Using Lemma 2.3 leads to

$$\begin{aligned}&\lim_{\rho\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} \varphi_\rho \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) \,\mathrm{d}x \\ &= \lim_{\rho\to 0} \int_{\mathbb{R}^N} \varphi_\rho d\omega = \omega(\{x_j\}) = \omega_j,\end{aligned} \tag{6.14}$$

and

$$\lim_{\rho\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} \varphi_\rho |u_n|^\alpha |v_n|^\beta \,\mathrm{d}x = \lim_{\rho\to 0} \int_{\mathbb{R}^N} \varphi_\rho d\nu = \nu(\{x_j\}) = \nu_j, \tag{6.15}$$

Summing up, from (6.9)–(6.15), taking the limit as $n \to \infty$, and then the limit as $\rho \to 0$, we infer that

$$\omega_j = 2\nu_j.$$

Step 2. We show that $\omega_\infty = 2\nu_\infty$, where ω_∞ and ν_∞ are given in Lemma 2.4. Let $\psi \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with $\psi \in [0,1]$, $\psi \equiv 0$ in $B_{1/2}(0)$ and $\psi \equiv 1$ in $\mathbb{R}^3 \setminus B_1(0)$. For any $R > 0$, we set

$$\psi_R(x) := \psi\left(\frac{x}{R}\right) = \begin{cases} 0, & |x| \le \frac{1}{2}R, \\ 1, & |x| \ge R. \end{cases}$$

By Lemma 6.4, we know that $\{\psi_R u_n\}_{n\in\mathbb{N}}$ and $\{\psi_R v_n\}_{n\in\mathbb{N}}$ are bounded in $H^1(\mathbb{R}^N)$. Then, we choose $(\psi_R u_n, \psi_R v_n)$ as a test function in (6.4). Furthermore, let $R \to \infty$, we can easily obtain that

$$\lim_{R\to\infty}\lim_{n\to\infty} \langle I'_\mu\left(u_n, v_n\right) + \lambda_{1,n}(u_n, 0) + \lambda_{2,n}(0, v_n), (\psi_R u_n, \psi_R v_n)\rangle = 0.$$

By the definition of ψ_R, we have

$$\int_{\{x\in\mathbb{R}^N\,:\,|x|>R\}} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) \,\mathrm{d}x \le \int_{\mathbb{R}^N} \psi_R \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) \,\mathrm{d}x$$

$$\leq \int_{\{x\in\mathbb{R}^N\,:\,|x|>\frac{1}{2}R\}} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) \mathrm{d}x.$$

Thus, by virtue of Lemma 2.4, we get

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} \psi_R(|\nabla u_n|^2 + |\nabla v_n|^2)\,\mathrm{d}x = \omega_\infty, \tag{6.16}$$

Similarly, we can deduce the following limits:

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} \psi_R|u_n|^\alpha|v_n|^\beta\,\mathrm{d}x = \lim_{R\to\infty}\int_{\mathbb{R}^N}\psi_R d\nu = \nu_\infty, \tag{6.17}$$

$$\begin{aligned}\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} \psi_R|u_n|^q\,\mathrm{d}x &= \lim_{R\to\infty}\int_{\mathbb{R}^N}\psi_R|u|^q\,\mathrm{d}x\\ &= \lim_{R\to\infty}\int_{|x|>\frac{1}{2}R}\psi_R|u|^q\,\mathrm{d}x = 0,\end{aligned} \tag{6.18}$$

$$\begin{aligned}\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} \psi_R|v_n|^q\,\mathrm{d}x &= \lim_{R\to\infty}\int_{\mathbb{R}^N}\psi_R|v|^q\,\mathrm{d}x\\ &= \lim_{R\to\infty}\int_{|x|>\frac{1}{2}R}\psi_R|v|^q\,\mathrm{d}x = 0.\end{aligned} \tag{6.19}$$

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} \lambda_{1,n}\psi_R u_n^2\,\mathrm{d}x = 0, \tag{6.20}$$

and

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} \lambda_{2,n}\psi_R v_n^2\,\mathrm{d}x = 0. \tag{6.21}$$

Then, from (6.16)–(6.21), taking the limit as $n\to\infty$, and then the limit as $R\to\infty$, we conclude that

$$\omega_\infty = 2\nu_\infty.$$

Step 3. We prove that $\nu_j = 0$ for any $j\in J$ and $\nu_\infty = 0$. Suppose by contradiction that there exists $j_0\in J$ such that $\nu_{j_0} > 0$ or $\nu_\infty > 0$. Steps 1 and 2 along with Lemmas 2.3 and 2.4 imply that

$$\nu_{j_0} \leq (S_{\alpha,\beta}^{-1}\omega_{j_0})^{\frac{\alpha+\beta}{2}} = S_{\alpha,\beta}^{-\frac{\alpha+\beta}{2}}(2\nu_{j_0})^{\frac{\alpha+\beta}{2}}, \tag{6.22}$$

or

$$\nu_\infty \leq (S_{\alpha,\beta}^{-1}\omega_\infty)^{\frac{\alpha+\beta}{2}} = S_{\alpha,\beta}^{-\frac{\alpha+\beta}{2}}(2\nu_\infty)^{\frac{\alpha+\beta}{2}}. \tag{6.23}$$

from which we find either $\nu_{j_0} \geq (\frac{S_{\alpha,\beta}}{2})^{\frac{N}{2}}$ or $\nu_\infty \geq (\frac{S_{\alpha,\beta}}{2})^{\frac{N}{2}}$. We first consider the case that (6.22) holds. It follows from Lemma 6.3 that there exists a positive constant μ^* large enough, such that

$$\gamma_\mu(a_1,a_2) < 2\left(\frac{1}{q\gamma_q} - \frac{1}{2^*}\right)\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}$$

for any $\mu\geq\mu^*$. Recalling $I_\mu(u_n,v_n)\to\gamma_\mu(a_1,a_2)$ and $P(u_n,v_n)=0$, we have

$$\begin{aligned}\gamma_\mu(a_1,a_2) &= \lim_{n\to\infty}\left(I_\mu(u_n,v_n) - \frac{1}{q\gamma_q}P(u_n,v_n)\right)\\ &\geq 2\left(\frac{1}{q\gamma_q} - \frac{1}{2^*}\right)\limsup_{n\to\infty}\int_{\mathbb{R}^N}\varphi_\rho|u_n|^\alpha|v_n|^\beta\,\mathrm{d}x\\ &\geq 2\left(\frac{1}{q\gamma_q} - \frac{1}{2^*}\right)\int_{\mathbb{R}^N}\varphi_\rho d\nu\\ &\geq 2\left(\frac{1}{q\gamma_q} - \frac{1}{2^*}\right)\nu_{j_0}\\ &\geq 2\left(\frac{1}{q\gamma_q} - \frac{1}{2^*}\right)\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}},\end{aligned}$$

in contradiction to the fact that

$$\gamma_\mu(a_1,a_2) < 2\left(\frac{1}{q\gamma_q} - \frac{1}{2^*}\right)\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}.$$

If (6.23) holds, the proof is similar to the above statement and we get a contradiction. Therefore, we have proved that $\nu_j = 0$ for any $j\in J$ and $\nu_\infty = 0$.

Moreover, by combining this with Lemmas 2.3 and 2.4, we have

$$\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \,dx \to \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \,dx, \text{ as } n \to \infty,$$

which completes the proof. □

Proof of Theorem 1.4. We note that all calculations above can be repeated similarly, replacing I_μ by the functional

$$\begin{aligned} I_\mu^+(u,v) = &\frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) dx - \frac{\mu}{q}\int_{\mathbb{R}^N} \left(|u^+|^q + |v^+|^q\right) dx \\ &- \frac{2}{2^*}\int_{\mathbb{R}^N} |u^+|^\alpha |v^+|^\beta \,dx. \end{aligned} \tag{6.24}$$

Using $(u^-,0)$ and $(0,v^-)$ as test functions in (6.24), in view of $(I_\mu^+)'(u,v)(u^-,0) = 0$ and $(I_\mu^+)'(u,v)(0,v^-) = 0$, where $w^- := \min\{w,0\}$, we get $u^- = 0, v^- = 0$, and so, $u \geq 0, v \geq 0$. Therefore, we have that $u_n^- \to 0, v_n^- \to 0$ a.e. in $\mathbb{R}^N$, as $n \to \infty$. Then it follows from Lemma 2.5 and Lemmas 6.4, 6.5 that the proof of Theorem 1.4 is finished. □

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Data availability

No data was used for the research described in the article.

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