Global a priori bounds and multiplicity results for quasilinear elliptic equations and inequalities

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Preface

The present work focuses on recent results of smooth and nonsmooth problems in terms of quasilinear elliptic equations and inequalities as well as certain a priori bounds for different classes of nonlinear problems. This thesis is divided into three chapters presenting the results of a series of research papers written in collaboration with G. Bonanno (Messina, Italy), S. El Manouni (Riyadh, Saudi Arabia), D. Motreanu (Perpignan, France), N. S. Papageorgiou (Athens, Greece), and R. Zacher (Halle and Ulm, Germany) during a period of about five years. In the specific chapters we will give a detailed description of the main results, the ideas, the techniques, the differences and the novelties to related works. The original papers are also attached to this work.

Chapter 1 concerns global a priori bounds for general classes of elliptic equations and inequalities by using different methods in the form of the so-called Moser and De Giorgi iteration. We apply these techniques to weak solutions of elliptic equations with a conormal derivative in both $W^{1,p}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ and give new L^∞-bounds for such solutions. In case of variational inequalities we present conditions on the sets of constraints guaranteeing the boundedness of their solutions.

Chapter 2 deals with quasilinear elliptic equations driven by nonlinear homogeneous and nonhomogeneous differential operators and with Dirichlet or Neumann boundary conditions. Based on critical point theory, comparison principles, truncation and perturbation techniques as well as Morse theory in the form of critical groups we establish new multiplicity results of such problems under various conditions of the right-hand perturbations being nonlinearities. Prototypes of our operators are the p-Laplacian ($(p-1)$-homogeneous) and the (p,q)-differential operator (nonhomogeneous).

Chapter 3 considers the question of how to deal with and transform methods from the smooth case to nonsmooth problems represented by variational inequalities involving generalized directional derivatives. Such inequalities are related to differential inclusions of Clarke's gradient type being the generalization of the classical derivative for locally Lipschitz continuous functions. The results presented in this chapter are mainly based on abstract critical point theorems partly developed in our papers.

The author wishes to express his gratitude to the co-authors stated above for their fruitful collaboration and many stimulating conversations. Of course, the author is greatly indebted to Professors Siegfried Carl (Halle, Germany) and Harry Yserentant (Berlin, Germany) for several helpful comments and for their support in general during the past years. Finally, the author gratefully acknowledges the DFG Research Center **Matheon** for their financial support.

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List of selected publications

The following selected manuscripts have been attached to this review paper. References to these papers are labeled by **C1**, **C2**, or **C3** for the corresponding chapter to which the paper is assigned, e.g. [Win10**C1**].

Win10**C1** P. Winkert, *L^∞-estimates for nonlinear elliptic Neumann boundary value problems*, NoDEA Nonlinear Differential Equations Appl. **17** (2010), no. 3, 289–302.

WZ12**C1** P. Winkert, R. Zacher, *A priori bounds for weak solutions to elliptic equations with nonstandard growth*, Discrete Contin. Dyn. Syst. Ser. S **5** (2012), no. 4, 865–878.

Win14**C1** P. Winkert, *On the boundedness of solutions to elliptic variational inequalities*, Set-Valued Var. Anal. **22** (2014), no. 4, 763–781.

MW11**C2** D. Motreanu, P. Winkert, *On the Fučik spectrum for the p-Laplacian with Robin boundary condition*, Nonlinear Anal. **74** (2011), no. 14, 4671–4681.

Win13**C2** P. Winkert, *Multiplicity results for a class of elliptic problems with nonlinear boundary condition*, Commun. Pure Appl. Anal. **12** (2013), no. 2, 785–802.

PW14b**C2** N. S. Papageorgiou, P. Winkert, *Resonant (p,2)-equations with concave terms*, Appl. Anal., accepted 2014.

PW14a**C2** N. S. Papageorgiou, P. Winkert, *On a parametric nonlinear Dirichlet problem with subdiffusive and equidiffusive reaction*, Adv. Nonlinear Stud. **14** (2014), no. 3, 747–773.

EMPW14**C2** S. El Manouni, N. S. Papageorgiou, P. Winkert, *Parametric nonlinear nonhomogeneous Neumann equations involving a nonhomogeneous differential operator*, Monatsh. Math., accepted 2014.

BMW11**C3** G. Bonanno, D. Motreanu, P. Winkert, *Variational-hemivariational inequalities with small perturbations of nonhomogeneous Neumann boundary conditions*, J. Math. Anal. Appl. **381** (2011), no. 2, 627–637.

BW14**C3** G. Bonanno, P. Winkert, *Multiplicity results to a class of variational-hemivariational inequalities*, Topol. Methods Nonlinear Anal. **43** (2014), no. 2, 493–516.

BMW13**C3** G. Bonanno, D. Motreanu, P. Winkert, *Boundary value problems with nonsmooth potential, constraints and parameters*, Dynam. Systems Appl. **22** (2013), no. 2-3, 385–396.

Chapter 1

Global a priori bounds

The main objective of this chapter is to present the results of the papers Winkert [Win10**C1**], Winkert [Win14**C1**], and Winkert-Zacher [WZ12**C1**] about global a priori bounds for classes of elliptic problems involving second-order differential operators that have a nonlinear boundary condition. Such estimates are important results in the theory of differential equations and often guarantee further knowledge about the smoothness of solutions. In this direction we mention the famous works of Lieberman [Lie88], [Lie91] stating the $C^{1,\alpha}$-regularity up to the boundary of solutions to partial differential equations of second order both with non-homogeneous Dirichlet- and nonlinear Neumann boundary condition. A crucial condition for the application of such results is the boundedness of the solutions.

In order to obtain such results, we make use of the so-called De Giorgi-Nash-Moser theory which provides iterative methods based on truncation techniques to get L^∞-bounds for certain equations. To be more precise, De Giorgi [DeG57] could show the Hölder continuity of weak solutions to elliptic equations of divergence form that have discontinuous coefficients while Nash [Nas58] derived the same results independently for both the elliptic and parabolic case. Afterwards, a new proof of De Giorgi's, resp. Nash's, theorem had been published by Moser [Mos60] by applying the Harnack inequality. He also gave a different proof of Nash's regularity result in the parabolic case by developing a parabolic version of the Harnack inequality, see [Mos64], [Mos67], and [Mos71]. More references with more historic exposition can be found in the habilitation thesis of Zacher [Zac11].

All in all, the techniques developed in the above-mentioned papers provided powerful tools to prove local and global boundedness, the Harnack and the weak Harnack inequality and the Hölder continuity of weak solutions. We refer to the monographs of Gilbarg-Trudinger [GT83], Ladyženskaja-Solonnikov-Ural′ceva [LSU68], and Lieberman [Lie96].

In our setting we distinguish between the Moser iteration and De Giorgi's iteration technique and apply these tools to different problems that have weak solutions in the usual Sobolev space $W^{1,p}(\Omega)$ and in the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ with p being a continuous function, respectively. Both techniques are modified versions of the original ones as can be found in the monographs of DiBenedetto [DiB93], Drábek-Kufner-Nicolosi [DKN97], and Ladyženskaja-Ural′tseva [LU68].

Note that the importance of our L^∞-estimates is twofold: First, we could show that the solutions (in the weak sense) of the considered equations are essentially bounded and thus belong to the space $L^\infty(\Omega)$. Such results for nonlinear equations with nonlinear boundary

condition are new and of independent interest. Indeed, only the boundedness of the solutions implies more regularity of the solutions, that is, they directly belong to $C^{1,\alpha}(\overline{\Omega})$ thanks to the regularity results of Lieberman [Lie88] provided the boundary function of the equation and the domain are sufficiently smooth. Second, as a direct consequence, the regularity is very important in the study of multiplicity results to such equations. In fact, several results that will be presented in Chapter 2 would not exist without the knowledge of the regularity of the solutions and so the content of this chapter plays an important role for further results, not only for those presented in this work.

Another novelty is the fact that it was possible to present similar L^∞-results for equations involving nonlinearities having nonstandard growth, that is, we are interested in the boundedness of weak solutions lying in the space $W^{1,p(\cdot)}(\Omega)$ with $p \in C(\overline{\Omega})$ satisfying $p > 1$ for $x \in \overline{\Omega}$. The difference to the constant exponent case is enormous and needs another approach. As the regularity results of Lieberman have been extended to the so-called variable exponent case (see Fan [Fan07]) we also obtain the $C^{1,\alpha}$-regularity in that case.

Finally, we combined the approaches mentioned above to suitable classes of variational inequalities with sets of constraints defined in both $W^{1,p}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$, respectively. We derived certain a priori estimates which are mainly dependent on the structure of the set of constraints. We will see that these sets need to be closed among others under specific operations like the pointwise maximum and minimum of Sobolev functions.

1.1 Elliptic equations with nonlinear boundary condition[1]

Given a bounded domain $\Omega \subseteq \mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ we consider the quasilinear, elliptic equation

$$Au = f(x, u, \nabla u) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = g(x, u) \quad \text{on } \partial\Omega, \tag{1.1.1}$$

where $\frac{\partial u}{\partial \nu}$ denotes the conormal derivative of u on $\partial\Omega$ and the mapping A is supposed to be a second-order quasilinear differential operator in divergence form of Leray-Lions type (see, for example, Carl-Le-Motreanu [CLM07]), that is

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)),$$

with $a_i \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ being Carathéodory functions that fulfill suitable p-structure assumptions for $1 < p < \infty$. In particular, the negative p-Laplacian defined by

$$-\Delta_p u = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \qquad 1 < p < \infty \tag{1.1.2}$$

fits in our setting. If $p = 2$, then $\Delta_p = \Delta$ becomes the well-known Laplace operator and we have $\Delta \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$ as the vector space of all bounded linear operators from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Moreover, the nonlinearities $f \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g \colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are some Carathéodory functions satisfying appropriate growth conditions with respect to $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, resp. $s \in \mathbb{R}$. The precise assumptions on the given data are stated on p. 49.

[1] Summary of the paper [Win10**C1**].

By means of the nonlinear Green's identity we say that $u \in W^{1,p}(\Omega)$ is a weak solution of equation (1.1.1) if it satisfies

$$\int_\Omega \sum_{i=1}^{N} a_i(x,u,\nabla u)\frac{\partial \varphi}{\partial x_i}dx = \int_\Omega f(x,u,\nabla u)\varphi dx + \int_{\partial\Omega} g(x,u)\varphi d\sigma, \tag{1.1.3}$$

for all test functions $\varphi \in W^{1,p}(\Omega)$. The main objective in [Win10**C1**] was the question of whether a weak solution of (1.1.1) is essentially bounded in Ω. We could give a positive answer.

Theorem 1.1.1 (Theorem 4.1 on p. 52) *Let the conditions (A1)–(A3),(F1)–(F3) and (G1)–(G3) stated on p. 49 be satisfied and let $u \in W^{1,p}(\Omega)$ be a solution of* (1.1.1)*. Then $u \in L^\infty(\Omega)$.*

The idea in our proof was to use the Moser iteration in its version presented in the monograph of Drábek-Kufner-Nicolosi [DKN97] who studied several quasilinear elliptic equations (weighted p-Laplacian problems) that have a homogeneous Dirichlet boundary condition to show existence and boundedness of weak solutions via Moser iteration. In the semilinear case we also mention the monograph of Ladyženskaja-Solonnikov-Ural′ceva [LSU68] adapting the method of De Giorgi and the work of Zhu [Zhu00]. Concerning a priori bounds for equations with homogeneous Neumann boundary conditions, we refer in the semilinear case to Zhu [Zhu99] (Ω is a bounded convex domain in $\mathbb{R}^3$), Wei-Xu [WX05] and in the quasilinear case to Motreanu-Motreanu-Papageorgiou [MMP09] as well as Hu-Papageorgiou [HP11] that both apply Moser's iteration. Note that the paper [HP11], which is a special case of [Win10**C1**], was submitted after acceptance of the paper [Win10**C1**]. The only paper considering problems with nonhomogeneous Neumann boundary condition is published by Lê [Lê06] about the eigenvalue problems of the p-Laplacian. Therein, the nonlinear boundary condition takes the form $|u|^{p-2}u$ which has an easy structure to deal with it meaning that Moser's iteration can be applied without difficulties. To the best of our knowledge the results in [Win10**C1**] were the first to provide the boundedness of weak solutions of problems like (1.1.1) involving a nonlinear boundary condition with a general growth condition and with a general operator possibly nonhomogeneous. In addition, the nonlinearity on the right-hand side of equation (1.1.1) may also depend on the gradient of the solution. The appearance of the nonlinear boundary term in (1.1.1), resp. in (1.1.3), was the main difficulty to overcome in our treatment. In fact we needed an inequality to estimate the boundary term by suitable integrals defined in the domain Ω (see (1.1.5) and the step from (1.1.4) to (1.1.6) below) which can in addition be controlled through scaling. Finally, we also refer to an interesting work of Pucci[2]-Servadei [PS08] about the regularity of weak solutions of homogeneous and inhomogeneous quasilinear elliptic equations. Although the nonlinearity on the right-hand side in [PS08] depends on the gradient, no growth condition to this variable is needed. The proof of the locally essentially boundedness of weak solutions works via Moser iteration.

Let us say something about the steps in the proof of Theorem 1.1.1. Since the space $W^{1,p}(\Omega)$ has a lattice structure, that is, the pointwise maximum and minimum of two functions of $W^{1,p}(\Omega)$ belongs again to the space, we can write every function $u \in W^{1,p}(\Omega)$ as $u = u_+ - u_-$ where $u_+ = \max(u,0)$ and $u_- = \max(-u,0)$ (at some places in this

[2]Patrizia Pucci (University of Perugia) sent me the paper after publishing the manuscript [Win10**C1**]. Since I did not know it before, it is not in the reference list of [Win10**C1**].

thesis we write u^+ and u^- instead). Therefore, we may assume, without loss of generality, that a weak solution of (1.1.1) is nonnegative. For $M > 0$ and $\kappa > 0$ we take $\varphi = (\min(u_+, M))^{\kappa p+1} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ as a test function in the weak formulation. Taking into account the structure conditions stated in (A1), (A3), (F3) as well as(G3) combined with an appropriate choice of Hölder's and Young's inequality we derive an estimate like

$$\left\| u_+^{(\kappa+1)p} \right\|_{W^{1,p}(\Omega)}^{(\kappa+1)p} \leq C_1 \left\| u_+^{(\kappa+1)p} \right\|_{L^p(\Omega)}^{(\kappa+1)p} + C_2 \left\| u_+^{(\kappa+1)p} \right\|_{L^p(\partial\Omega)}^{(\kappa+1)p} + C_3 \tag{1.1.4}$$

with positive constants C_1, C_2, C_3 depending on the parameter κ. In order to estimate the boundary norm on the right-hand side of (1.1.4) through norms of $W^{1,p}(\Omega)$ and $L^p(\Omega)$ we make use of suitable embeddings and interpolation results concerning Besov, Lizorkin-Triebel, and Sobolev Slobodeckij spaces. Indeed, we have the continuous embedding

$$B_{p,p}^{\frac{1}{p}+\tilde{\varepsilon}}(\Omega) \to L^p(\partial\Omega),$$

where $\tilde{\varepsilon} > 0$ is chosen such that $\frac{1}{p} + \tilde{\varepsilon} < 1$ guaranteeing that a Lipschitz boundary $\partial\Omega$ is sufficient while $B_{p,p}^s, s \in (0,1)$, denotes the Besov space coinciding with the Sobolev Slobodeckij space $W^{s,p}$. Since $B_{p,p}^{\frac{1}{p}+\tilde{\varepsilon}}(\Omega)$ lies between the spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$ we may apply real interpolation to obtain

$$\|v\|_{L^p(\partial\Omega)} \leq \|v\|_{W^{1,p}(\Omega)}^{\frac{1}{p}+\tilde{\varepsilon}} \|v\|_{L^p(\Omega)}^{1-\frac{1}{p}-\tilde{\varepsilon}} \quad \text{for all } v \in W^{1,p}(\Omega), \tag{1.1.5}$$

(see the monographs of Runst-Sickel [RS96] and Triebel [Tri78], [Tri83], [Tri92]). By means of (1.1.4) and (1.1.5) along with Young's inequality as well as the Sobolev embedding theorem it can be shown that

$$\left\| u^+ \right\|_{L^{(\kappa+1)p^*}(\Omega)} \leq C_3^{\frac{1}{\kappa+1}} C_4^{\frac{1}{\sqrt{\kappa+1}}} C_5^{\frac{1}{(\kappa+1)p}} \left[\int_\Omega u_+^{(\kappa+1)p} dx + 1 \right]^{\frac{1}{(\kappa+1)p}} \tag{1.1.6}$$

with some $C_3, C_4, C_5 > 0$. Now, we can start the typical bootstrap arguments to prove first that $u_+ \in L^r(\Omega)$ for any $r \in (1, \infty)$ through a suitable choice of the parameter κ and afterwards we prove the uniform estimate with respect to κ. In the end this yields the $L^{r_n}(\Omega)$-norm of u_+ on the left-hand side and a finite norm of u_+ on the right-hand side independent of r_n while r_n is a sequence converging to $+\infty$. Therefore, u_+ is essentially bounded in Ω by a constant depending on u_+. The same can be done for u_- which yields the boundedness of u by a constant depending on the structure conditions and the solution itself[3]. We point out that the monotonicity condition (A2) is not needed in the proof of Theorem 1.1.1, but in general it is necessary to ensure the existence of solutions to equation (1.1.1).

As mentioned in the beginning of this section, a direct consequence of Theorem 1.1.1 is the $C^{1,\alpha}$-regularity of weak solutions to equation (1.1.1) for $A = -\Delta_p$ being the negative p-Laplacian (see (1.1.2)). Indeed, supposing that the boundary function $g\colon \partial\Omega \times \mathbb{R} \to \Omega$ is Hölder continuous on $\partial\Omega \times [-M_0, M_0], M_0 > 0$, (see p. 58), we may apply Theorem 2 of Lieberman [Lie88] to obtain that every solution $u \in W^{1,p}(\Omega)$ of (1.1.1) belongs to $C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1]$.

[3]Note that in the final published version of [Win10**C1**] (see p. 290, line 11 from the top) the phrase "independent of" has to be replaced by "depending on". The journal forgot to correct the misprint during the process of the galley proof. It is clear that the constant cannot be independent of all possible solutions.

The L^∞-bounds obtained in [Win10**C1**] are in general very useful and of independent interest. In fact, the existence of multiple solutions including sign-changing solutions is mainly based on regularity results which follows from Theorem 1.1.1 (see, for example, Papageorgiou-Rădulescu [PR14], [PR14b], Winkert [Win10a], [Win10b], and the results in Chapter 2).

1.2 Equations with nonlinearities having nonstandard growth[4]

In Sect. 1.1 we have seen that a weak solution $u \in W^{1,p}(\Omega)$ of (1.1.1) is essentially bounded in Ω and thus belongs to $L^\infty(\Omega)$. Now, we may generalize these results to weak solutions $u \in W^{1,p(\cdot)}(\Omega)$ where p is a given continuous function acting on $\overline{\Omega}$. In contrast to Sect. 1.1 we will see that Moser's iteration is less suitable for equations defined in so-called variable exponent spaces. To be more precise, it is not clear for the author how to derive similar global a priori bounds as in Sect. 1.1 to problems with nonstandard growth via Moser's iteration since the gradient of the special test function would be very complicated as p occurs in the exponent of the test function. Here, the idea is to use a modified version of De Giorgi's iteration technique.

As before, let Ω be a bounded domain in $\mathbb{R}^N$, $N > 1$, with Lipschitz boundary $\Gamma := \partial\Omega$ and let $p \in C(\overline{\Omega})$ be a function that satisfies $1 < p^- := \inf_{x\in\overline{\Omega}} p(x)$. We deal with elliptic equations of the form

$$-\operatorname{div} \mathcal{A}(x, u, \nabla u) = \mathcal{B}(x, u, \nabla u) \quad \text{in } \Omega, \qquad \mathcal{A}(x, u, \nabla u) \cdot \nu = \mathcal{C}(x, u) \quad \text{on } \Gamma, \quad (1.2.1)$$

where $\nu(x)$ denotes the outer unit normal of Ω at $x \in \Gamma$ and the Carathéodory functions $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ satisfy suitable $p(x)$-structure conditions, see hypotheses (H) on p. 62. These conditions are very general and incorporate a large number of differential operators. Of course, the so-called $p(x)$-Laplacian defined by

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u),$$

fits in our setting which reduces to the standard p-Laplacian if $p(x) \equiv p$.

In recent years there has been a growing interest in the study of elliptic problems with a $p(x)$-structure, which are also termed problems with nonstandard growth conditions. Equations of this type appear in the study of non-Newtonian fluids with thermo-convective effects (see Antontsev-Rodrigues [AR06], Zhikov [Zhi97]), electro-rheological fluids (see Diening [Die02], Rajagopal-Růžička [RR01], Růžička [Růž00]), the thermistor problem (see Zhikov [Zhi97b]), or the problem of image recovery (see Chen-Levine-Rao [CLR06]).

Since p is a continuous function on $\overline{\Omega}$ being strictly greater than one, we easily verify that $p^- > 1$ and $p^+ < \infty$ where $p^- := \min_{x\in\overline{\Omega}} p(x)$ and $p^+ := \max_{x\in\overline{\Omega}} p(x)$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u \;\middle|\; u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)} dx < +\infty \right\}$$

[4] Summary of the paper [WZ12**C1**].

equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\tau > 0 : \int_\Omega \left|\frac{u(x)}{\tau}\right|^{p(x)} dx \le 1\right\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by $W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\}$ with the norm $\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{p(\cdot)}(\Omega)}$. Both spaces are reflexive Banach spaces. For more information and basic properties of variable exponent spaces we refer to the papers of Fan-Zhao [FZ01], Kováčik-Rákosník [KR91] and the monograph of Diening-Harjulehto-Hästö-Růžička [DHHR11].

A function $u \in W^{1,p(\cdot)}(\Omega)$ is said to be a weak solution (subsolution, supersolution) of equation (1.2.1) if

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi dx = (\le,\ge)\int_\Omega \mathcal{B}(x,u,\nabla u)\varphi dx + \int_\Gamma \mathcal{C}(x,u)\varphi d\sigma, \tag{1.2.2}$$

holds for all nonnegative test functions $\varphi \in W^{1,p(\cdot)}(\Omega)$, where $d\sigma$ denotes the usual $(N-1)$-dimensional surface measure.

Recall $y_+ = \max(y,0)$ the main result reads as follows.

Theorem 1.2.1 (Theorem 1.1 on p. 62) *Let the assumptions (H) stated on p. 62 be satisfied. Then there exist positive constants $C = C(p, q_0, q_1, a_3, a_4, a_5, b_0, b_1, b_2, c_0, c_1, N, \Omega)$ and $\alpha = \alpha(p, q_0, q_1)$ such that the following assertions hold.*

(i) If $u \in W^{1,p(\cdot)}(\Omega)$ is a weak subsolution of (1.2.1), then

$$\operatorname*{ess\,sup}_\Omega u \le 2\max\left(1, C\left[\int_\Omega u_+^{q_0(x)}dx + \int_\Gamma u_+^{q_1(x)}d\sigma\right]^\alpha\right).$$

(ii) If $u \in W^{1,p(\cdot)}(\Omega)$ is a weak supersolution of (1.2.1), then

$$\operatorname*{ess\,inf}_\Omega u \ge -2\max\left(1, C\left[\int_\Omega (-u)_+^{q_0(x)}dx + \int_\Gamma (-u)_+^{q_1(x)}d\sigma\right]^\alpha\right).$$

Note that the exponents $q_0 \in C(\overline{\Omega})$ and $q_1 \in C(\Gamma)$ are chosen such that $p(x) \le q_0(x) < p^*(x)$ for $x \in \overline{\Omega}$ and $p(x) \le q_1(x) < p_*(x)$ for $x \in \Gamma$ with the critical exponents $p^*(x)$ and $p_*(x)$. Therefore, due to the compact embeddings $W^{1,p(\cdot)}(\Omega) \to L^{q_0(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega) \to L^{q_1(\cdot)}(\Gamma)$, the right-hand sides in Theorem 1.2.1 (i),(ii) are finite. The coefficients a_i, b_l and c_k came directly from the structure conditions.

Since a weak solution of (1.2.1) is both a weak subsolution and a weak supersolution of (1.2.1) we directly obtain that every weak solution of (1.2.1) is essentially bounded in Ω and the estimates in (i) and (ii) of Theorem 1.2.1 are valid. In contrast to Sect. 1.1 we could not only generalize the results to problems with nonstandard growth but could also give precise lower and upper bounds describing the dependence on the given data. In addition to De Giorgi's iteration method, as mentioned in the beginning of this chapter, we also used the localization method combined with an appropriate partition of unity. In other words, the idea in the proof is to handle problems involving variable exponents as problems with constant exponents which is possible due to the continuity of the exponents p, q_0, and q_1. In our approach we do not need a log-Hölder continuity condition as it is normally used. Such

condition is a main ingredient in the theory of variable exponent spaces because it guarantees among others the density of smooth functions. In this direction we refer to the overview article of Harjulehto-Hästö-Lê-Nuortio [HHLN10] which contains the most famous results and corresponding references in the theory of variable exponent spaces from its beginning in 1931 until 2010. The complete results including their proofs can be found in the monograph of Diening-Harjulehto-Hästö-Růžička [DHHR11].

The motivation of the paper [WZ12**C1**] came not only from the urge to generalize the results in [Win10**C1**] (see Sect. 1.1), but also from the results of Fan-Zhao [FZ99] who considered regularity of weak solutions to the equation

$$-\operatorname{div}\mathcal{A}(x,u,\nabla u)=\mathcal{B}(x,u,\nabla u),\quad x\in\Omega, \tag{1.2.3}$$

where $\mathcal{A},\mathcal{B}$ satisfy the same structure conditions as in [WZ12**C1**, p. 866]. In Theorem 4.1 of [FZ99] it is shown that every weak solution of (1.2.3) belongs to $L^{\infty}_{\mathrm{loc}}(\Omega)$ and additionally, if the weak solution is bounded on the boundary, then it belongs to $L^{\infty}(\Omega)$. Of course, such result can be applied to problems like (1.2.3), which has a homogeneous Dirichlet boundary condition, but it fails if we have a conormal derivative boundary condition as in (1.2.1). Unfortunately, some authors applied the results of Fan-Zhao to nonlinear Neumann problems which is in fact not allowed, see, for example, Dai [Dai13]. We could fill, however, this gap in the literature.

Boundedness of weak solutions to problem (1.2.1) in the case of the negative $p(x)$-Laplacian, that is $-\operatorname{div}\mathcal{A}(x,u,\nabla u)=-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ and with a homogeneous Neumann boundary condition has also been obtained by Gasiński-Papageorgiou [GP11] by applying a modified version of Moser's iteration under the very strong condition that the exponent p is continuously differentiable on $\overline{\Omega}$. Concerning boundedness and regularity results for problems of type (1.2.3), in particular for the special case

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)=0,$$

we further refer to Acerbi-Mingione [AM01], [AM02], Antontsev-Consiglieri [AC09] , Chiadò Piat-Coscia [CPC97], Diening-Ettwein-Růžička [DER07], Eleuteri-Habermann [EH08], Fan [Fan07], [Fan10], Fan-Zhao [FZ00], Habermann-Zatorska-Goldstein [HZG08], Harjulehto-Kinnunen-Lukkari [HKL07], Liskevich-Skrypnik [LS10], Lukkari [Luk10], [Luk09] and the references given therein.

The proof of Theorem 1.2.1 can be summarized as follows. First we derive truncated energy estimates for weak subsolutions and weak supersolutions by applying the structure conditions stated in hypotheses (H) on p. 62. To be more precise, taking a weak subsolution $u\in W^{1,p(\cdot)}(\Omega)$ of (1.2.1) and choosing $\varphi=(u-k)_+\in W^{1,p(\cdot)}(\Omega)$ with a free parameter $k\geq 1$ as test function, we obtain

$$\int_{\{x\in\Omega:u(x)>k\}}|\nabla u|^{p(x)}dx\leq d_1\int_{\{x\in\Omega:u(x)>k\}}u^{q_0(x)}dx+d_2\int_{\{x\in\Gamma:u(x)>k\}}u^{q_1(x)}d\sigma, \tag{1.2.4}$$

with positive constants d_1,d_2 depending on the given structure conditions (see Lemma 3.1 on p.65). Note that the choice of the test function is typically in the usage of De Giorgi's iteration (see, for example, DiBenedetto [DiB93]) and in contrast to Moser's iteration presented in Sect. 1.1 we easily verify that the test function is now more simple – because p is a function depending on x, we have no problems with complicated gradients as it would be the case by applying Moser's iteration. We also mention that we utilized the fact that the

variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ has lattice structure, see Le [Le09]. A similar result as (1.2.4) for weak supersolutions of (1.2.1) is shown in Lemma 3.2 on p. 66.

In the next step we used the localization method combined with a suitable choice of a partition of unity. Since $\overline{\Omega}$ is compact, there exists, for any $R > 0$, a finite open cover $\{B_i(R)\}_{i=1,\dots,m}$ of balls $B_i := B_i(R)$ with radius R such that $\overline{\Omega} \subset \bigcup_{i=1}^m B_i(R)$. Moreover, since $p \in C(\overline{\Omega})$, $q_0 \in C(\overline{\Omega})$, and $q_1 \in C(\Gamma)$, these functions are uniformly continuous on $\overline{\Omega}$ and Γ, respectively. Recalling that

$$p(x) \le q_0(x) < p^*(x), \quad x \in \overline{\Omega}, \quad \text{and} \quad p(x) \le q_1(x) < p_*(x), \quad x \in \Gamma,$$

we may take $R > 0$ small enough such that

$$p_i^+ \le q_{0,i}^+ < (p_i^-)^*, \qquad p_i^+ \le q_{1,i}^+ < (p_i^-)_*, \qquad i = 1, \dots, m,$$

where

$$p_i^+ = \max_{x \in B_i \cap \overline{\Omega}} p(x), \quad q_{0,i}^+ = \max_{x \in B_i \cap \overline{\Omega}} q_0(x), \quad p_i^- = \min_{x \in B_i \cap \overline{\Omega}} p(x), \quad q_{1,i}^+ = \max_{x \in B_i \cap \Gamma} q_1(x)$$

and $(p_i^-)^*, (p_i^-)_*$ denote the corresponding critical exponents. Next, we choose a partition of unity $\{\xi_i\}_{i=1}^m \subset C_0^\infty(\mathbb{R}^N)$ associated to the open cover $\{B_i(R)\}_{i=1,\dots,m}$, that is, we have

$$\operatorname{supp} \xi_i \subset B_i, \quad 0 \le \xi_i \le 1, \quad i = 1, \dots, m, \quad \text{and} \quad \sum_{i=1}^m \xi_i = 1 \ \text{ on } \overline{\Omega}.$$

Based on these observations using Hölders inequality, suitable embedding results, and (1.2.4) we obtain an iterative inequality which allows us to apply a lemma concerning the geometric convergences of sequences of numbers developed by Vergara-Zacher [VZ10], based on the original version published in Ladyženskaja-Solonnikov-Ural′ceva [LSU68, Chapter II, Lemma 5.6] (see also DiBenedetto [DiB93, Chapter I,Lemma 4.1]). These steps have only been taken for weak subsolutions. The case for weak supersolutions works similarly and so Theorem 1.2.1 is proved.

An interesting open problem is the question of whether the results in [WZ12**C1**] remain valid in the case that p is no longer continuous but essentially bounded. The continuity of p was a significant hypothesis by applying the localization method and it is not clear how to deal with problems involving nonlinearities that have nonstandard growth concerning boundedness of solutions in this case. In general we cannot suppose that $p^+ < (p^-)^*$ and so it is not possible to use embedding results for constant exponent Sobolev spaces.

The parabolic counterpart to [WZ12**C1**] is currently in process [WZ], that is, we are interested in a priori bounds to weak solutions of the equation

$$\begin{aligned} u_t - \operatorname{div} \mathcal{A}(t,x,u,\nabla u) &= \mathcal{B}(t,x,u,\nabla u) && \text{in } (0,T) \times \Omega, \\ \mathcal{A}(t,x,u,\nabla u) \cdot \nu &= \mathcal{C}(t,x,u) && \text{on } (0,T) \times \Gamma, \\ u(0,x) &= u_0(x) && \text{in } \Omega, \end{aligned}$$

with $u_0 \in L^2(\Omega)$, $T > 0$, and with an exponent p now depending on t and x. It becomes apparent that the continuity of p will not be sufficient and we require a log-Hölder condition on p. One of the main difficulties is to find a suitable equivalent weak formulation – analog to the constant exponent case via Steklov average (see DiBenedetto [DiB93]).

Finally, we mention that the truncation methods of Moser and De Giorgi presented in this

chapter cannot be applied to equations of a higher order because the corresponding Sobolev spaces do not satisfy a lattice structure. In this case we have to break fresh ground.

1.3 Boundedness of solutions to variational inequalities[5]

In Sections 1.1 and 1.2 we have seen which methods can be used to prove a priori bounds for quasilinear elliptic equations with standard and nonstandard growth. This section is now concerned with the question of whether these methods can be transferred to variational inequalities and if so which further conditions are needed. First, it is not surprising that a priori bounds or regularity results of variational inequalities in general are mainly dependent on the structure of the set of constraints. Therefore, our aim is to find suitable assumptions on the sets of constraints in order to apply the iterative techniques presented in this chapter so far.

Let us first describe the problem under consideration. Recall that $\Omega \subset \mathbb{R}^N$, $N > 1$, is again a bounded domain with Lipschitz boundary $\partial\Omega$, we consider the following inequality: Find $u \in K$ such that

$$\begin{aligned} &\int_\Omega A(x,u,\nabla u)\cdot\nabla(v-u)dx + \int_\Omega F(x,u,\nabla u)(v-u)dx \\ &+ \int_\Omega j_1^\circ(x,u;v-u)dx + \int_{\partial\Omega} j_2^\circ(x,u;v-u)d\sigma \geq 0, \quad \text{for all } v \in K, \end{aligned} \tag{1.3.1}$$

where K is a subset of a Banach space V (will be specified below) and $j_k^\circ(x,s;r)$ $(k=1,2)$ denotes the generalized directional derivative of a locally Lipschitz function $s \mapsto j_k(x,s)$ at s in the direction r. Moreover, the maps $A\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ and $F\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ are supposed to be Carathéodory functions satisfying suitable structure conditions stated in (H1) on p. 80 and in (H2) on p. 87. The aim of the treatment in [Win14**C1**] was to present conditions on the set of constraints, namely K, such that every solution $u \in K$ of (1.3.1) is essentially bounded in Ω. In order to specify the space V, we discussed two different cases: In the first case we chose $V = W^{1,p}(\Omega)$ with $1 < p < \infty$ and in the second one we set $V = W^{1,p(\cdot)}(\Omega)$ with $p \in C(\overline{\Omega})$ and $1 < \inf_{\overline{\Omega}} p$. These cases will be handled by different methods, the first one via Moser iteration and the second via De Giorgi's iteration technique similar to Sects. 1.1 and 1.2. Note that we do not suppose that the set K is closed and convex in V. But in general, this is the typical assumption in the existence theory of inequalities like (1.3.1).

Inequalities of type (1.3.1) are called variational-hemivariational inequalities due to the presence of generalized directional derivatives $j_k^\circ(x,s;r)$ of the locally Lipschitz functions $s \mapsto j_k(x,s)$ at s in the direction r given by

$$j_k^\circ(x;s;r) := \limsup_{y\to s, t\to 0^+} \frac{j_k(x,y+tr)-j_k(x,y)}{t},$$

which occurs in the associated definition of generalized gradients defined by

$$\partial j_k(x,s) = \{\xi \in \mathbb{R} : j_k^\circ(x,s;r) \geq \xi r, \forall r \in \mathbb{R}\}, \quad k=1,2.$$

[5] Summary of the paper [Win14**C1**].

A first mathematical treatment of generalized directional derivatives and generalized gradients has been developed by Clarke [Cla75] which was extended by himself in the monographs Clarke [Cla83], [Cla90]. This is the reason why a generalized gradient is often called Clarke's gradient. We also refer to the work of Chang [Cha81] who gave new aspects to the theory of Clarke's gradient in both an abstract and a concrete setting. Note that this gradient is indeed a generalization because it coincides with the usual derivative $\big(j_k(x,\cdot)\big)'$ if $j_k(x,\cdot)$ is smooth, and if $j_k(x,\cdot)$ is convex and nonsmooth, then it represents the well-known subdifferential of $j_k(x,\cdot)$ in the sense of convex analysis. In the eighties, Panagiotopoulos [Pan85] was one of the first who recognized the large field of applications of such inequalities, which describe, for example, problems in mechanics and engineering governed by nonconvex, possibly nonsmooth energy functionals called superpotentials. Theses kinds of energy functionals appear if nonmonotone, possibly multivalued constitutive laws are taken into account. We refer to the monographs of Carl-Le-Motreanu [CLM07], Motreanu-Rădulescu [MR03], Naniewicz-Panagiotopoulos [NP95], and Panagiotopoulos [Pan93] to complete the picture of (variational-)hemivariational inequalities.

Since it is often asked what the difference between a hemivariational and a variational-hemivariational inequality is, we will give a short classification of these classes.

(i) If $j_k = 0$ $(k = 1, 2)$, then inequality (1.3.1) reduces to a classical variational inequality whose treatment is (more or less) well known. We refer to the monograph of Kinderlehrer-Stampacchia [KS80] finding the basic existence results to inequalities of this kind.

(ii) If $K = V$ is the whole space and j_k $(k = 1, 2)$ nonsmooth (as it is given), then inequality (1.3.1) is called a hemivariational inequality. Note that generalized directional derivatives are only positively homogeneous with respect to the direction and so it continues as a inequality even if we test it on the whole space. We also point out that hemivariational inequalities contain as a special case differential inclusions of the form

$$\begin{aligned} -\operatorname{div} A(x,u,\nabla u) + F(x,u,\nabla u) + \partial j_1(x,u) &\ni 0 \qquad \text{in } \Omega, \\ A(x,u,\nabla u)\cdot\nu + \partial j_2(x,u) &\ni 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

(iii) If $K = V$ and j_k $(k = 1, 2)$ are smooth, then inequality (1.3.1) becomes

$$\int_\Omega A(x,u,\nabla u)\cdot\nabla v\,dx + \int_\Omega F(x,u,\nabla u)v\,dx + \int_\Omega j_1'(x,u)v\,dx + \int_{\partial\Omega} j_2'(x,u)v\,d\sigma = 0,$$

for all $v \in V$ which means that $u \in V$ is a weak solution to the problem

$$-\operatorname{div} A(x,u,\nabla u) + F(x,u,\nabla u) + j_1'(x,u) = 0 \text{ in } \Omega,\ A(x,u,\nabla u)\cdot\nu + j_2'(x,u) = 0 \text{ on } \partial\Omega.$$

What this all amounts to is that if a term of a generalized directional derivative appears, then we have a hemivariational inequality and if in addition the set of constraints is a strict subset of the whole space not being a subspace, then we speak of a variational-hemivariational inequality. We also note that there is a close relationship between variational-hemivariational inequalities and so-called multi-valued variational inequalities. Under certain assumptions (for example, monotonicity conditions) both problems are equivalent, see Carl [Car11].

Let us turn back to the results presented in [Win14**C1**]. In the first part we are interested in how to obtain L^∞-estimates for inequality (1.3.1) if $V = W^{1,p}(\Omega)$ is the usual Sobolev space for $1 < p < \infty$. The idea in our approach is to use Moser's iteration in its form

presented in Sect. 1.1. To do this, we establish conditions on the sets of constraints which allow test functions in (1.3.1) as required. To be more precise we suppose the following.

(K1) For $u \in K$, $\alpha > 0$, and $\kappa > 0$ there exists $t > 0$ such that $\varphi = u - t v_\alpha^{\kappa p} u \in K$.

(K2) For $u \in K$, $\beta > 0$, and $\kappa > 0$ there exists $h > 0$ such that $\psi = u - h(-v_\beta)^{\kappa p} u \in K$.

Here, $v_\alpha(x) := \min(u_+(x), \alpha)$ and $v_\beta(x) := \max(u_-(x), -\beta)$ which both belong to $W^{1,p}(\Omega)$. Of course, our hypotheses on K are applicable to a number of sets of interest. We mention the following ones

$$K = \{u \in W^{1,p}(\Omega) : u \le \vartheta \quad \text{a.e. in } \Omega\}, K = \{u \in W^{1,p}(\Omega) : u \ge -\vartheta \quad \text{a.e. in } \Omega\},$$
$$K = \{u \in W^{1,p}(\Omega) : \omega \le u \le \psi \quad \text{a.e. in } \Omega\}, K = \{u \in W^{1,p}(\Omega) : |\nabla u| \le \vartheta \quad \text{a.e. in } \Omega\}$$

with given functions $\vartheta, \omega, \psi \colon \Omega \to \mathbb{R}$ such that $\vartheta \ge 0, \omega \le 0 \le \psi$ a.e. in Ω.

The main result in the first part reads as follows.

Theorem 1.3.1 (Theorem 3.1 on p. 81) *Let hypothesis (H1) stated on p. 80 be satisfied and let $u \in K$ be a solution of* (1.3.1). *Then there exists a constant $C_1 > 0$ such that the following assertions hold.*

(1) If condition (K1) is satisfied, then ess $\sup_{x\in\Omega} u(x) \le C_1$.

(2) If condition (K2) is satisfied, then ess $\inf_{x\in\Omega} u(x) \ge -C_1$.

The proof of Theorem 1.3.1 is mainly based on the conditions (K1), (K2) in combination with the properties of generalized directional derivatives in order to estimate these terms in a suitable way and finally, since our conditions are more general than those in [Win10**C1**], we develop a multiplicative inequality to estimate the boundary integrals by appropriate integrals defined in the domain. Indeed, we prove the following proposition.

Proposition 1.3.2 (Proposition 2.1 on p. 79) *Let $\Omega \subset \mathbb{R}^N, N > 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$, let $1 < p < \infty$, and let q be such that $p \le q < p_*$ with the critical exponent $p_* = \frac{(N-1)p}{N-p}$ if $p < N$ and $p_* = \infty$ if $p \ge N$. Then, for every $\varepsilon > 0$, there exist constants $a_1 > 0$ and $a_2 > 0$ such that*

$$\|u\|^p_{L^q(\partial\Omega)} \le \varepsilon \|u\|^p_{W^{1,p}(\Omega)} + a_1 \varepsilon^{-a_2} \|u\|^p_{L^p(\Omega)} \qquad \text{for all } u \in W^{1,p}(\Omega). \tag{1.3.2}$$

Although the statement in Proposition 1.3.2 is intuitive and not new, we did not find a reference for it in the form presented here. Indeed, if we replace $a_1\varepsilon^{-a_2}$ by a constant $C(\varepsilon)$, then one can prove inequality (1.3.2) indirectly via contradiction[6] (see, for example, Khademloo [Kha11, Lemma 1]) but generally without knowledge of the concrete structure of the constant $C(\varepsilon)$. Since in our proof we choose ε dependent on the parameter which will later be sent to ∞ we have to control the term $C(\varepsilon)$ and so the statement in (1.3.2) is absolutely necessary in our case. The only similar result we found to Proposition 1.3.2 is stated in the monograph of Maz'ja [Maz85, Chapter 1, Sect. 1.4.7, Corollary 2].

The results of Theorem 1.3.1 were mainly motivated by a paper of Kovalevsky-Nicolosi [KN99] who gave a priori bounds for classical variational inequalities by applying a modified version of Moser's iteration. It should be mentioned that we could improve their work in

[6] Rüdiger Landes (University of Oklahoma) first told me at the 8th AIMS Conference in Dresden in 2010 about the proof. Later, he sent me a very elegant simple proof for this inequality via e-mail.

several ways. On one hand we extend their results to variational-hemivariational inequalities including nonlinear boundary terms and on the other hand we have weaker assumptions on the set of constraints. Indeed, hypothesis 2.6(i) in [KN99] is not needed in our treatment. For the sake of convenience we do not use Sobolev spaces with weights, but this case can be done in the same way. Another novelty of [Win14**C1**] is the treatment of variational-hemivariational inequalities restricted to sets K belonging to variable exponent spaces $W^{1,p(\cdot)}(\Omega)$ with p continuous on $\overline{\Omega}$ (see below). A priori estimates for inequalities of this kind have not been published before.

As already mentioned, the second part of [Win14**C1**] is concerned with the case $V = W^{1,p(\cdot)}(\Omega)$ for $p(x) > 1$ for all $x \in \overline{\Omega}$. Here, we will apply the results of [WZ12**C1**] (see Sect. 1.2) based on De Giorgi's iteration technique to inequalities like (1.3.1). In contrast to the case of constant exponent Sobolev spaces, our assumptions on the sets of constraints seem now more restrictive. We suppose the following.

(K3) For $u \in K$ and $\kappa > 0$ there exists $t > 0$ such that $\varphi = u - t(u - \kappa)_+ \in K$.

(K4) For $u \in K$ and $\kappa > 0$ there exists $h > 0$ such that $\psi = u - h(u + \kappa)_- \in K$.

These conditions are deduced in a natural way from the choice of the test functions presented in Sect. 1.2. For instance, if $\vartheta \colon \Omega \to \mathbb{R}$ is a given nonnegative function, the set

$$K = \left\{ v \in W^{1,p(\cdot)}(\Omega) : |\nabla v| \leq \vartheta \quad \text{a.e. in } \Omega \right\}$$

satisfies the assumptions stated in (K3) and (K4). The main result in this part is similar to Theorem 1.3.1. In fact, we have the following theorem.

Theorem 1.3.3 (Theorem 4.2 on p. 90) *Assume hypothesis (H2) stated on p. 87 and let $u \in K$ be a solution of* (1.3.1). *Then there exists a constant $C_2 > 0$ such that the following assertions hold.*

(1) If condition (K3) is satisfied, then $\operatorname{ess\,sup}_{x\in\Omega} u(x) \leq C_2$.

(2) If condition (K4) is satisfied, then $\operatorname{ess\,inf}_{x\in\Omega} u(x) \geq -C_2$.

In order to prove Theorem 1.3.3 we can mainly use the results of Sect. 1.2. Indeed, it is enough to prove truncation energy estimates as in (1.2.4) via the structure conditions and hypotheses on the set of constraints. Such inequalities are also known as (global) Caccioppoli inequalities. Then, [WZ12**C1**, Theorem 1.1] could be used in the same way to obtain the results above. We also give a lemma including necessary conditions on the sets of constraints in order to satisfy the conditions in (K3), (K4). An interesting example is the cone of nonnegative elements of $W^{1,p(\cdot)}(\Omega)$, that is

$$K_{\geq 0} = \{ v \in W^{1,p(\cdot)}(\Omega) : v \geq 0 \quad \text{a.e. in } \Omega \}.$$

Since $K_{\geq 0}$ has lattice structure and contains the positive constant functions we see, for $u \in K$ and $\kappa > 0$, that

$$\min(u, \kappa) = u - (u - \kappa)_+ \in K_{\geq 0}.$$

Therefore (K3) is fulfilled and Theorem 1.3.3(1) can be applied to obtain that every solution $u \in K_{\geq 0}$ of (1.3.1) is essentially bounded by above in Ω. Of course, the assertion in Theorem 1.3.3(2) is clearly satisfied through the definition of $K_{\geq 0}$.

Chapter 2

Multiplicity results for quasilinear elliptic boundary value problems

This second chapter is devoted to the existence of multiple solutions[1] to quasilinear elliptic equations involving homogeneous or nonhomogeneous operators. We will present the main results and ideas of the papers Motreanu-Winkert [MW11**C2**], Winkert [Win13**C2**], Papageorgiou-Winkert [PW14b**C2**], [PW14a**C2**], and El Manouni-Papageorgiou-Winkert [EMPW14**C2**].

In the past two decades multiplicity results of semilinear and quasilinear elliptic equations have been extensively studied by various authors under different assumptions. A large interest in the study of such problems is not only the existence of multiple solutions but also their qualitative properties, such as regularity properties and sign information. In particular, the question of the sign information has been taking center in the last years. While in the beginning the existence of positive solutions (in general solutions of constant sign) took a relevant part of the investigation, it is nowadays common to prove (if possible) the existence of sign-changing solutions (also known as nodal solutions) as well.

Our results make use of variational methods based on critical point theory combined with suitable truncation and perturbation techniques along with Morse theory in terms of critical groups. In recent years equations involving nonhomogeneous operators have been of growing interest and their treatment exhibits some difficulties in contrast to homogeneous operator equations. A prototype of such an operator is the (p, q)-differential operator being the sum of the negative p-Laplacian and the negative q-Laplacian with $1 < q < p$. Of course, the negative p-Laplacian is a $(p-1)$-homogeneous operator which can be easily seen from its representation, see (1.1.2).

Finally, we mention some recent works of the author concerning nonhomogeneous operators being related to the topic of this chapter but that are not part of this thesis. A complete study of equations involving superlinear nonlinearities can be found in [PWa], new existence results to Robin problems are obtained in [PWb], and finally, a bifurcation-type result for Dirichlet problems has been recently finished in [PWc]. An overview about some recent results can be found in the book chapter [MW14].

[1] Under "multiple solutions" we understand the existence of at least two nontrivial solutions to a given equation.

2.1 The Robin Fučík spectrum of the p-Laplacian[2]

The first notation of the Fučík spectrum being a generalization of the usual spectrum was done by Fučík [Fuč76] and Dancer [Dan76/77] in the seventies. Precisely, given a bounded domain $\Omega \subset \mathbb{R}^N$ and considering a self-adjoint linear operator T on $L^2(\Omega)$ with compact resolvent and eigenvalues $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$., the so-called Fučík spectrum[3] Σ of T is defined as the set of all pairs $(a, b) \in \mathbb{R}^2$ such that the equation

$$Tu = au^+ - bu^-$$

has a nontrivial solution where $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$ stand for the positive and negative part of u. Fučík [Fuč76] and Dancer [Dan76/77] were one of the first authors who recognized that the set Σ plays an important part in the study of semilinear equations of type

$$Tu = g(x, u),$$

involving a Carathéodory function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ being a jumping nonlinearity, that is, the limits

$$a = \lim_{s \to +\infty} \frac{g(x, s)}{s}, \quad b = \lim_{s \to -\infty} \frac{g(x, s)}{s}$$

exist and $a \neq b$. If $a = b = \infty$, the function g is called a rapid nonlinearity (see the monograph of Fučík [Fuč80, p. 9]).

In [Fuč76], Fučík considered the case of a semilinear ordinary differential equation, that is, in the case of the negative Laplacian in one-dimension. He proved that the Fučík spectrum Σ_2 is composed of two families of curves in $\mathbb{R}^2$ emanating from the points (λ_k, λ_k) determined by the eigenvalues λ_k of the negative Laplacian in one-dimension. Dancer [Dan76/77] was able to prove that the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ are isolated in Σ_2. Afterwards, many authors studied the Fučík spectrum Σ_2 for the negative Laplacian $-\Delta$ with Dirichlet boundary condition on a bounded domain $\Omega \subset \mathbb{R}^N$ (see Arias-Campos [AC95], Các [Các89], Dancer [Dan93], Margulies-Margulies [MM97], Marino-Micheletti-Pistoia [MMP94], Micheletti [Mic94], Micheletti-Pistoia [MP95], Pistoia [Pis97], Schechter [Sch94], and the references therein). We also mention the work of De Figueiredo-Gossez[4] [DFG94] who constructed a first nontrivial curve in Σ_2 through (λ_2, λ_2) and characterized it variationally. For $p \neq 2$ and in one dimension, Drábek [Drá92] has shown that Σ_p has similar properties to the linear case, that is $p = 2$.

Let us comment on some relevant works concerning the Fučík spectrum of the p-Laplacian in the general case $1 < p < \infty$ and for an arbitrary bounded domain $\Omega \subset \mathbb{R}^N, N \geq 1$. A starting point was the work done by Cuesta-de Figueiredo-Gossez [CDFG99] who treated the Fučík spectrum Σ_p of the negative p-Laplacian with homogeneous Dirichlet boundary conditions. They constructed a first nontrivial curve through the point (λ_2, λ_2) asymptotic

[2] Summary of the paper [MW11**C2**].

[3] The name is due to the Czech mathematician Svatopluk Fučík (21th October 1944 – 18th May 1979). In some references the Fučík spectrum is also called Dancer-Fučík spectrum, see, for example, the monograph of Zhang [Zha13, Chapter 7].

[4] In 2010 Jean-Pierre Gossez (University of Brussels) sent me some relevant references concerning the Fučík spectrum which were very helpful in deriving the results in [MW11**C2**].

to $\mathbb{R}\times\lambda_1$ and $\lambda_1\times\mathbb{R}$ at infinity and characterized it variationally by a mountain-pass approach. Shortly afterwards Arias-Campos-Gossez [ACG00] considered the Fučik spectrum Θ_p of the negative p-Laplacian with a homogeneous Neumann boundary condition (see also Alif [Ali02] and Massa [Mas04]) proving similar results as in the Dirichlet case. One important difference is the fact that the asymptotic behavior of the first nontrivial curve needs a different approach for the cases $p \leq N$ and $p > N$. The existence of an unbounded sequence of curves in Σ_p has been obtained by Micheletti-Pistoia [MP01] in case $p \geq 2$ and shortly after by Perera [Per04] in the general case $p > 1$. Finally, we emphasize the paper [MR04] of Martínez-Rossi who considered the Fučik spectrum $\widetilde{\Sigma}_p$ of the p-Laplacian according to a Steklov boundary condition. Similar to [CDFG99] and [ACG00] it was constructed a first nontrivial curve in $\widetilde{\Sigma}_p$ through the point (λ_2, λ_2) where now λ_2 stands for the second eigenvalue of the Steklov eigenvalue problem of the p-Laplacian.

Based on these works mentioned above, the motivation in the paper [MW11**C2**] was to study the Fučík spectrum of the p-Laplacian with a Robin boundary condition, that is, we were looking for the set $\widehat{\Sigma}_p$ of those points $(a, b) \in \mathbb{R}^2$ such that

$$-\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1} \quad \text{in } \Omega, \qquad |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = -\beta|u|^{p-2}u \quad \text{on } \partial\Omega, \tag{2.1.1}$$

has a nontrivial (weak) solution with a parameter β belonging to $[0, +\infty)$. Note that the Fučik spectrum Θ_p is incorporated in problem (2.1.1) by taking $\beta = 0$. If $a = b = \lambda$, problem (2.1.1) is reduced to the Robin eigenvalue problem for the p-Laplacian, that is

$$-\Delta_p u = \lambda|u|^{p-2}u \quad \text{in } \Omega, \qquad |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = -\beta|u|^{p-2}u \quad \text{on } \partial\Omega. \tag{2.1.2}$$

The treatment of problem (2.1.2) is known and was discussed in the work of Lê [Lê06] about the eigenvalue problems of the p-Laplacian. More precisely we know that the first eigenvalue λ_1 of problem (2.1.2) is simple, isolated and can be characterized as

$$\lambda_1 = \inf_{u\in W^{1,p}(\Omega)} \left\{ \|\nabla u\|^p_{L^p(\Omega)} + \beta\|u\|^p_{L^p(\partial\Omega)} : \|u\|^p_{L^p(\Omega)} = 1 \right\}.$$

Denoting φ_1 as the first eigenfunction associated with the first eigenvalue λ_1 of (2.1.2), we further know that φ_1 has constant sign and due to the regularity theory of Lieberman [Lie88] and the strong maximum principle of Vázquez [Váz84] it belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ and satisfies $\varphi_1 > 0$ in $\overline{\Omega}$. It should be noted that every eigenfunction of (2.1.2) corresponding to an eigenvalue $\lambda > \lambda_1$ must change sign in Ω.

The results in [MW11**C2**] are mainly based on a variational approach relying on the corresponding energy functional $J\colon W^{1,p}(\Omega) \to \mathbb{R}$ to problem (2.1.1) which is given by

$$J(u) = \int_\Omega |\nabla u|^p dx + \beta\int_{\partial\Omega} |u|^p d\sigma - \int_\Omega \left(a(u^+)^p + b(u^-)^p\right) dx.$$

Obviously, $J \in C^1(W^{1,p}(\Omega))$ and the critical points of J are exactly the weak solutions of problem (2.1.1). Note that in comparison with the associated functionals related to the Fučík spectrum for the Dirichlet and Steklov problems, the functional J does not incorporate the usual norm of the space $W^{1,p}(\Omega)$ and in contrast to the Neumann problem it has an additional boundary term.

Based on a mountain-pass procedure and by applying various ideas and techniques on the basis of [CDFG99], [ACG00], and [MR04] we could show in Theorem 3.3 on p. 101 that the

Fučík spectrum $\widehat{\Sigma}_p$ contains a first nontrivial curve denoted by $\mathcal{C}$ which can be given by

$$\mathcal{C} = \{(s + c(s), c(s)), (c(s), s + c(s)) : s \geq 0\},$$

where

$$c(s) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,+1]} \tilde{J}_s(u), \quad \Gamma = \{\gamma \in C([-1,1], S) : \gamma(-1) = -\varphi_1 \text{ and } \gamma(1) = \varphi_1\}.$$

Here, $\tilde{J}_s$ denotes the restriction of the C^1-functional $J_s \colon W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J_s(u) = \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\sigma - s \int_\Omega (u^+)^p dx$$

to the C^1-submanifold $S = \left\{u \in W^{1,p}(\Omega) : \int_\Omega |u|^p dx = 1\right\}$ of $W^{1,p}(\Omega)$. A direct consequence of the representation of $\mathcal{C}$ is a variational characterization of the second eigenvalue λ_2 of the Robin eigenvalue problem stated in (2.1.2). Indeed, setting $s = 0$ yields $c(0) = \lambda_2$ and so we have

$$\lambda_2 = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,1]} \left[\int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\sigma\right],$$

(see Corollary 3.4 on p. 102). Furthermore, it was possible to prove that the curve $\mathcal{C}$ is Lipschitz continuous and decreasing (see Proposition 4.2 on p. 102) while the asymptotic behavior needs again, similar to the Neumann and Steklov cases, a different approach for $p \leq N$ and $p > N$. In fact, if $p \leq N$, one has $\lim_{s \to +\infty} c(s) = \lambda_1$ (see Theorem 4.3 on p. 103). More difficult is the case $p > N$. First, we can assume that $\beta > 0$ because the case $\beta = 0$ is included in [ACG00]. Then, the main idea was to work with the norm

$$\|u\|_\beta = \|\nabla u\|_{L^p(\Omega)} + \beta \|u\|_{L^p(\partial\Omega)}, \tag{2.1.3}$$

which is an equivalent norm on $W^{1,p}(\Omega)$ (see, for example, Deng [Den09, Theorem 2.1]). Based on this, one obtains that the limit of $c(s)$ as $s \to +\infty$ is

$$\overline{\lambda} = \inf_{u \in L} \max_{r \in \mathbb{R}} \frac{\int_\Omega |\nabla(r\varphi_1 + u)|^p dx + \beta \int_{\partial\Omega} |r\varphi_1 + u|^p d\sigma}{\int_\Omega |r\varphi_1 + u|^p dx},$$

where $L = \{u \in W^{1,p}(\Omega) : u \text{ vanishes somewhere in } \overline{\Omega}, u \not\equiv 0\}$. Moreover, there holds $\overline{\lambda} > \lambda_1$. This result is stated in Theorem 4.4 on p. 103.

In order to have a detailed picture of the Fučík spectrum of the p-Laplacian with the different boundary conditions mentioned above, we refer to the overview chapter of Motreanu-Winkert [MW12] which gives the main results of all spectra, pointing out their differences and difficulties in order to proof these facts. Figure 2.1 visualizes how the first nontrivial curve $\mathcal{C}$ looks.

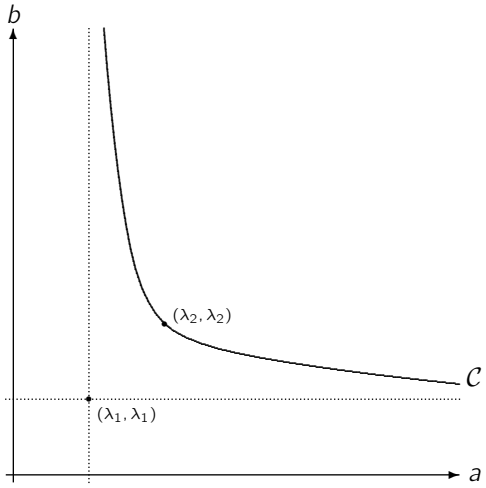

Figure 2.1: The first nontrivial curve $\mathcal{C}$ of the Fučík spectrum $\widehat{\Sigma}_p$.

2.2 Equations involving the p-Laplace differential operator[5]

In this section we want to present the applicability of the results of Sect. 2.1 in order to obtain multiple solutions for p-Laplace equations with a nonlinear Neumann boundary condition. Consider the perturbed Robin Fučík spectrum

$$\begin{aligned} -\Delta_p u &= a(u^+)^{p-1} - b(u^-)^{p-1} + f(x,u) && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= h(x,u) - \theta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{2.2.1}$$

with Carathéodory nonlinearities $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$, $h\colon \partial\Omega\times\mathbb{R}\to\mathbb{R}$ and parameters a, b, θ to be specified. The question is how to derive multiple solutions to problem (2.2.1) dependent on the parameters. By multiple solutions we mean here the existence of at least three nontrivial solutions including complete sign information for these solutions. Precisely, one solution is positive, one is negative, and finally, one solution has a changing sign. To show this we suppose that both f and h are bounded on bounded sets, $(p-1)$-sublinear near zero, and f fulfills a $(p-1)$-superlinearity condition near infinity while h satisfies a much weaker form of this. In addition, the boundary function h has to satisfy a locally Hölder condition near zero in order to apply the regularity results of Lieberman [Lie88]. The parameters in (2.2.1) are chosen such that $0<\theta<\beta$ and the point $(a,b)\in\mathbb{R}^2_+$ is above the first nontrivial curve $\mathcal{C}$ of the Robin Fučík spectrum $\widehat{\Sigma}_p$ presented in Sect. 2.1 with fixed $\beta>0$ (see Figure 2.2). Of course, if $a=b=\lambda$, then the last condition reduces to $\lambda>\lambda_2$ where λ_2 is the second eigenvalue of the Robin eigenvalue problem given in (2.1.2) (see Figure 2.3). In this case, problem (2.2.1) becomes a perturbed Robin eigenvalue problem with $\theta>0$.

[5]Summary of the paper [Win13**C2**].

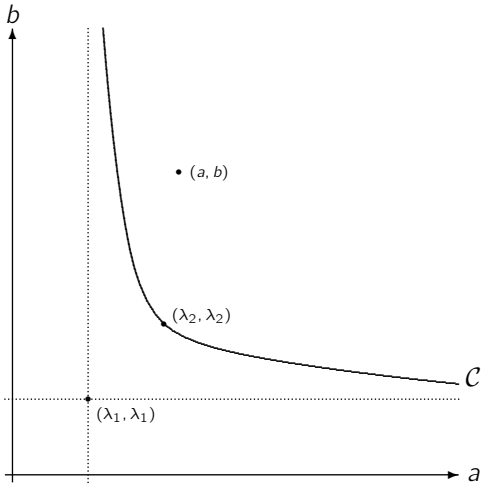

Figure 2.2: The point (a, b) is above the curve $\mathcal{C}$.

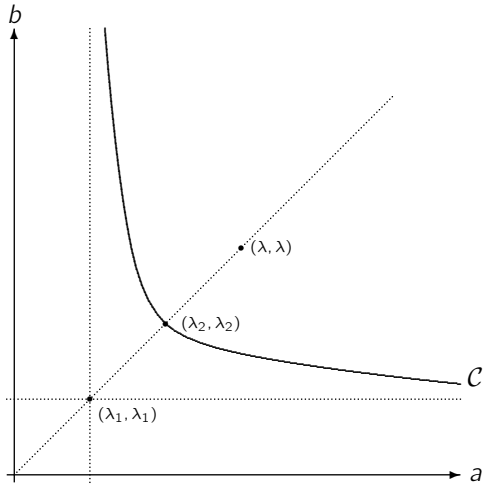

Figure 2.3: The number $\lambda = a = b$ is greater than λ_2.

In our setting we do not need differentiability, polynomial growth or some integral conditions on the mappings f and h. To the best of our knowledge this is the first work where the solutions obtained depend directly on the Robin Fučík spectrum of the negative p-Laplacian (see Sect. 2.1). In addition, our assumptions are very natural and no sign changing condition on f near the origin is needed as is required in some other works of the author for problems of this kind, see, for example, Winkert [Win10a], [Win11], [Win10c]. The proofs of the main results rely mainly on variational and topological tools, for instance, critical point theory, the mountain-pass theorem, the second deformation lemma and the properties of the Robin Fučík spectrum of the negative p-Laplacian (see Sect. 2.1).

Problems involving the p-Laplacian with a nonhomogeneous, nonlinear boundary condition have been studied by various authors concerning multiple solutions in recent years. We refer, without guarantee of completeness, to the reference list of the paper [Win13**C2**] stating the main works in this direction. The first paper which gives not only multiple solutions but also a sign changing solution dependent on a Fučík spectrum is that of Carl-Perera [CP02] who studied a perturbed Dirichlet Fučík spectrum on a bounded domain.

First, we proved the existence of constant sign solutions to problem (2.2.1).

Theorem 2.2.1 (Theorem 4.1 on p. 115) *Let the conditions (H1)–(H7) stated on pp. 107–108 be satisfied and let* $0 < \theta < \beta$. *Then, for every* $a > \lambda_1$ *and* $b \in \mathbb{R}$, *there exists a smallest positive weak solution* $u_+ = u_+(a) \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ *of* (2.2.1) *in the order interval* $[0, \vartheta_a e]$ *while for every* $b > \lambda_1$ *and* $a \in \mathbb{R}$, *there exists a greatest negative weak solution* $u_- = u_-(b) \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ *within* $[-\vartheta_b e, 0]$.

Note that Theorem 2.2.1 implies even the existence of extremal constant sign solutions to problem (2.2.1) which means that we have a smallest positive and a greatest negative solution to equation (2.2.1). The idea in the proof of Theorem 2.2.1 is to show the existence of two ordered pairs of sub- and supersolutions given by the scaled first eigenfunction φ_1 of the Robin eigenvalue problem stated in (2.1.2) and by a scaled function e which arises as the unique solution of a suitable chosen auxiliary problem. Taking into account the method of

sub- and supersolution (see, for example, Carl [Car06]) to these pairs we directly conclude the existence of two constant sign solutions – one with positive and one with negative sign. A difficult part is to prove that we indeed have extremal constant sign solutions. Here, we used, among others, the fact that every eigenfunction corresponding to an eigenvalue $\lambda > \lambda_1$ of the Robin eigenvalue problem must change sign in Ω. In order to have the regularity of the solutions obtained we first used the results of Sect. 1.1 (or Sect. 1.2) to have the boundedness which allows us to use the $C^{1,\alpha}$-regularity of Lieberman [Lie88] and Vazquez's strong maximum principle [Váz84].

The idea to find a sign changing solution is simple but the implementation is absolutely nontrivial. If we find a solution between the extremal constant sign solutions obtained in Theorem 2.2.1 which is unequal to these extremal solutions and unequal to the zero function, then it must be a sign changing solution. To realize this we first consider suitable energy functionals concerning problem (2.2.1) involving appropriate truncation functions to make sure that the coercivity and the Palais-Smale condition is satisfied. Here, we used again the equivalent norm on $W^{1,p}(\Omega)$ given in (2.1.3) (with θ instead of β). Furthermore, in [Win13**C2**, Lemma 5.1 and Lemma 5.2] we proved a relationship between their critical points and weak solutions to (2.2.1) and it is shown that our extremal constant sign solutions obtained in Theorem 2.2.1 are local minimizers of the truncated energy functional $J_0 \colon W^{1,p}(\Omega) \to R$ given by

$$\begin{aligned} J_0(u) =& \frac{1}{p}\left(\|\nabla u\|^p_{L^p(\Omega)} + \theta\|u\|^p_{L^p(\partial\Omega)}\right) - \int_\Omega \int_0^{u(x)} \left(a\tau_+(x,s)^{p-1} - b|\tau_-(x,s)|^{p-1}\right) ds dx \\ & - \int_\Omega \int_0^{u(x)} f(x,\tau_0(x,s)) ds dx - \int_{\partial\Omega} \int_0^{u(x)} h\left(x, \tau_0^{\partial\Omega}(x,s)\right) ds d\mu, \end{aligned}$$

where $\tau_0 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $\tau_0^{\partial\Omega} \colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ cut-off from above by u_+ and below by u_-. Next we may apply the mountain-pass theorem to J_0 and the elements u_+, u_- which gives a critical point $u_0 \in W^{1,p}(\Omega)$ of J_0 lying between u_+ and u_- implying that it is indeed a weak solution of (2.2.1). Since $J_0(u_+) \neq J_0(u_0) \neq J_0(u_-)$ (comes directly from the mountain-pass theorem) we know that u_0 is a sign-changing solution if it is not identically zero. Showing this is the main part and needs the properties of the curve $\mathcal{C}$ constructed in Sect. 2.1 (recall the assumptions on the point (a, b)), the hypotheses on f, h, and finally the usage of the second deformation lemma. To be more precise, we show the existence of $\widetilde{\gamma} \in \Gamma_W$ where Γ_W arises from Γ (see Sect. 2.1) by replacing S through $W^{1,p}(\Omega)$ and $-\varphi_1, \varphi_1$ by u_-, u_+ (see Theorem 2.2.1) such that

$$J_0(\widetilde{\gamma}(t)) \neq 0, \quad \text{for all } t \in [-1, 1]. \tag{2.2.2}$$

In other words, $\widetilde{\gamma}$ is a continuous path on $[-1, 1]$ with values in $W^{1,p}(\Omega)$ connecting the smallest positive and the greatest negative solution of (2.2.1). The validity of (2.2.2) is done in three steps that in the end connect three paths ($\gamma_-, \varepsilon\gamma_C$ and γ_+, $\varepsilon > 0$ small) to have the required path, see Figure 2.4. To construct the paths γ_+ and γ_- we make use of the second deformation lemma.

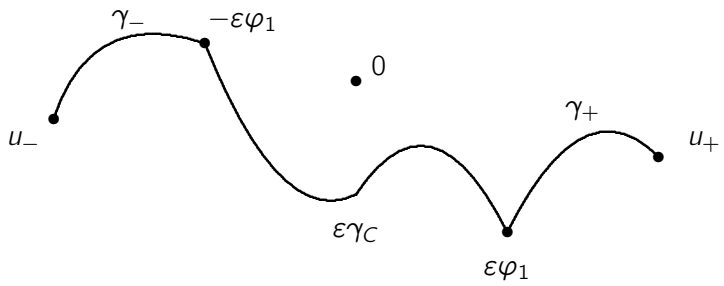

Figure 2.4: The union of the curves $\gamma_-, \varepsilon\gamma_C$ and γ_+ yields γ_W (provided ε is sufficiently small) with the required properties.

This finally leads to the following existence theorem about a nodal solution.

Theorem 2.2.2 (Theorem 5.3 on p. 120) *Under the hypotheses (H) and (H8) stated on pp. 107, 108, and 111 problem* (2.2.1) *possesses a nontrivial sign-changing weak solution* $u_0 \in C^1(\overline{\Omega})$.

2.3 Resonant $(p, 2)$-equations with concave terms[6]

The main objective of this section is the study of the parametric equation

$$-\Delta_p u - \Delta u = f(x, u) - \lambda|u|^{q-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{2.3.1}$$

with numbers $1 < q < 2 < p < \infty$, a Carathéodory nonlinearity $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$, and a nonhomogeneous operator $-\Delta_p - \Delta$ being the sum of the negative p-Laplacian and the negative Laplacian. Such operators are called $(p, 2)$-differential operators and arise, for example, in quantum physics in connection with Derrick's model [Der64] concerning the existence of solitons which was investigated by Benci-D'Avenia-Fortunato-Pisani [BDFP00] and in plasma physics, see, for example, Cherfils-Il'yasov [CI05].

Since $q \in (1, 2)$ we see that the right-hand side of equation (2.3.1) contains a parametric concave term with a negative sign. A starting point in the study of elliptic equations involving concave nonlinearities was initiated by Ambrosetti-Brezis-Cerami [ABC94] in the semilinear case with a nonlinearity

$$f(s) = \lambda|s|^{q-2}s + |s|^{r-2}s, \quad 1 < q < 2 < r < 2^*. \tag{2.3.2}$$

Therefore, we have in (2.3.2) a competing effect of a concave and a convex term. The novelty of our work [PW14b**C2**] is the combination of a nonhomogeneous operator driven by a $(p, 2)$-differential operator and the fact that our parametric concave term has a negative sign. In the reference list of [PW14b**C2**] one finds several works in this direction but mostly with a positive sign. It should be mentioned that the negative sign produced a different geometry of the problem and thus needed a completely different approach.

The main goal of this work was to show the existence of five nontrivial solutions to equation (2.3.1) for a sufficiently small parameter $\lambda > 0$. In the first part we could show the

[6] Summary of the paper [PW14b**C2**].

existence of four nontrivial constant sign solutions under the hypotheses H_1 stated on p. 130. Note that condition H_1(ii) implies that resonance can occur asymptotically at $\pm\infty$ with respect to the principal eigenvalue $\lambda_1(p) > 0$ of the negative p-Laplacian with a homogeneous Dirichlet boundary condition. We also mention that assumption H_1(iv), which says that $f(x,\cdot)$ is locally lower Lipschitz for a.a. $x \in \Omega$, is of technical structure, but we need this condition to apply the tangency principle stated in Pucci-Serrin [PS07]. Our first result reads as follows.

Theorem 2.3.1 (Proposition 3.5 on p. 136) *Let hypotheses H_1 stated on pp. 130–131 be satisfied and let $\lambda \in (0, \lambda^*)$. Then problem* (2.3.1) *admits at least four nontrivial solutions of constant sign, namely*

$$u_0, \hat{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\} \quad \text{and} \quad v_0, \hat{v} \in -\left(C_0^1(\overline{\Omega})_+\right) \setminus \{0\},$$

such that

$$u_0(x), \hat{u}(x) > 0 \quad \text{for all } x \in \Omega \quad \text{and} \quad v_0(x), \hat{v}(x) < 0 \quad \text{for all } x \in \Omega.$$

Moreover, u_0 and v_0 are local minimizers of φ_λ.

The number $\lambda^* > 0$ is sufficiently small and occurs for the first time in Proposition 3.3 of this paper. Since our parametric concave term has a negative sign, it was not possible to prove that the constant sign solutions of Theorem 2.3.1 belong to $\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ (resp. $-\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$). This can be easily verified by viewing the assumptions of the strong maximum principle in Pucci-Serrin [PS07].

The proceeding of the proof can be summarized as follows. First we consider the corresponding energy functional $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ and the positive/negative truncated functionals $\varphi_\lambda^\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$. It is shown that these functionals are coercive and then fulfill the Palais-Smale condition[7]. The first positive and negative solutions came via the direct method from the corresponding truncated functionals. To prove their nontriviality we had to keep the parameter $\lambda > 0$ small and so we find a number $\lambda^* > 0$ as stated in the theorem. The second constant sign solutions arose from the mountain-pass theorem and the fact that $u = 0$ is a local minimizer of our three functionals. In contrast to the application of the mountain-pass theorem in Sect. 2.2 the obtained critical point can be more easily identified as a nontrivial one. We do not need to construct a path as complicated as before.

As written in the beginning the aim was a five-solutions-theorem, but how to find a fifth solution? We need to strengthen the hypotheses on the nonlinearity $f : \Omega \times \Omega \to \Omega$ meaning that $f(x,\cdot)$ must now be a C^1-function on $\mathbb{R}$. The precise hypotheses are stated in H_2 on p. 139. The main idea is now to show the applicability of an abstract critical point result due to Perera [Per97] which says the following.

Theorem 2.3.2 (Theorem 3.1 in [Per97]) *Let $X = X_1 \oplus X_2$ be a Banach space with $0 < k := \dim X_1 < \infty$. Suppose that $I \in C^1(X, \mathbb{R})$ satisfies*

(I_1) there exists $\rho > 0$ such that $\sup_{S_\rho^1} I < 0$, where $S_\rho^1 := \{u \in X_1 : \|u\| = \rho\}$,

(I_2) $I \geq 0$ on X_2, and

[7] In general, this implication is not true, but in our setting we could give a positive answer.

(I_3) there exists a nonzero vector $e \in X_1$ such that I is bounded below on the half-space $\{se + u_2 : s \geq 0, u_2 \in X_2\}$.

In addition, assume that I satisfies the Palais-Smale condition and has only isolated critical values with each critical value corresponding to a finite number of critical points. Then, I has two (different) critical points u_1, u_2 with $I(u_1) < 0 \leq I(u_2)$ and $C_{k-1}(I, u_1) \neq 0, C_k(I, u_2) \neq 0$.

Here, $C_{k-1}(I, u_1)$ stands for the $(k-1)$-th critical group of I at the isolated point u_1. Such critical groups are defined by relative singular homology groups and have a growing importance in recent years in the study of multiple solutions to quasilinear elliptic equations like (2.3.1). In our setting we take $X = W_0^{1,p}(\Omega)$ and

$$V_m = \bigoplus_{i=1}^{m} E\left(\hat{\lambda}_i(2)\right) \quad \text{and} \quad W_m = W_0^{1,p}(\Omega) \cap V_m^{\perp},$$

which implies $W_0^{1,p}(\Omega) = V_m \bigoplus W_m$ and $d_m = \dim V_m < \infty$. By $E\left(\hat{\lambda}_i(2)\right)$ we denote the eigenspace of the i-th eigenvalue of the Laplacian with a homogeneous Dirichlet boundary condition. After proving the assumptions of Theorem 2.3.2 applied to the functional $I = \varphi_\lambda$ we obtain a critical point y_0 (u_1 in the theorem) which can be shown as unequal to the constant sign solutions stated in Theorem 2.3.1 and unequal zero. Therefore, we have found a fifth nontrivial solution to equation (2.3.1). In fact, Theorem 2.3.2 gives us two different critical points but in the case of u_2 we do not know if it is possibly the zero function or if it coincides with the constant sign solutions $\hat{u}, \hat{v}$ obtained via the mountain-pass theorem (see Theorem 2.3.1). Maybe, there is a sixth nontrivial solution of problem (2.3.1).

The complete multiplicity theorem including five nontrivial solutions is stated in Theorem 4.3 on p. 141 whereby two solutions have a positive and two have a negative sign. Unfortunately, it was not possible to prove the sign of the fifth solution obtained from Theorem 2.3.2. This is again due to the negative sign occurring in the parametric concave term on the right-hand side of (2.3.1). It remains an interesting open problem if y_0 is a sign changing solution.

2.4 General nonhomogeneous operators[8]

In Sect. 2.3 we considered equations involving $(p, 2)$-differential operators being nonhomogeneous where the special structure of the operator was a main feature. Now, we bring our focus on equations that have general nonhomogeneous operators of divergence form not necessarily of $(p, 2)$-structure and with different boundary conditions. Let us consider the boundary value problem

$$-\operatorname{div} a(\nabla u) = g(x, u) \quad \text{in } \Omega, \qquad Bu = 0 \quad \text{on } \partial\Omega, \tag{2.4.1}$$

with a right-hand side nonlinearity $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ being of Carathéodory type and a boundary operator B, for example, the Dirichlet or the Neumann one. Later on we will specify the concrete structure of g and B. First, we want to introduce our assumptions on $a\colon \mathbb{R}^N \to \mathbb{R}^N$.

[8] Summary of the papers [PW14a**C2**] and [EMPW14**C2**].

To do so, let $\vartheta \in C^1(0,+\infty)$ be a function satisfying

$$0 < \hat{c} \leq \frac{t\vartheta'(t)}{\vartheta(t)} \leq c_0 \quad \text{and} \quad c_1 t^{p-1} \leq \vartheta(t) \leq c_2\left(1+t^{p-1}\right), \tag{2.4.2}$$

for all $t > 0$ and with some constants $\hat{c}, c_0, c_1, c_2 > 0$. The hypotheses on $a(\cdot)$ read as follows.

H(a): $a(\xi) = a_0(\|\xi\|)\xi$ for all $\xi \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,\infty)$, $t \mapsto t a_0(t)$ is strictly increasing, $\lim_{t\to 0^+} t a_0(t) = 0$, $\lim\limits_{t\to 0^+} \dfrac{t a_0'(t)}{a_0(t)} > -1$;

(ii) $\|\nabla a(\xi)\| \leq c_3 \dfrac{\vartheta(\|\xi\|)}{\|\xi\|}$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and some $c_3 > 0$;

(iii) $(\nabla a(\xi) y, y)_{\mathbb{R}^N} \geq \dfrac{\vartheta(\|\xi\|)}{\|\xi\|}\|y\|^2$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and all $y \in \mathbb{R}^N$.

Based on these conditions we can easily see that $a \in C^1\left(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^N\right) \cap C\left(\mathbb{R}^N, \mathbb{R}^N\right)$ and that $\xi \to a(\xi)$ is maximal monotone and strictly monotone. If $G_0(t) = \int_0^t s a_0(s) ds$ for all $t > 0$ and if $G\colon \mathbb{R}^N \to \mathbb{R}$ is defined by $G(\xi) = G_0(\|\xi\|)$ for all $\xi \in \mathbb{R}^N$, then $G(\cdot)$ is the primitive of $a(\cdot)$ and $\xi \to G(\xi)$ is convex with $G(0) = 0$. Therefore we may apply critical point theory to problems like (2.4.1). For a detailed explanation of all consequences we refer directly to the paper [PW14a**C2**]. Note that the assumptions in H(a)(i) are required to apply the strong maximum principle to such general operators due to Pucci-Serrin [PS07].

Our conditions incorporate several differential operators of interest. Although being $(p-1)$-homogeneous, we first mention the p-Laplace differential operator by setting $a(\xi) = \|\xi\|^{p-2}\xi$ with $1 < p < \infty$ which gives (1.1.2) and the potential is $G(\xi) = \frac{1}{p}\|\xi\|^p$ for all $\xi \in \mathbb{R}^N$. A prototype of a nonhomogeneous differential operator is the (p,q)-differential operator which arises by $a(\xi) = \|\xi\|^{p-2}\xi + \|\xi\|^{q-2}\xi$ with $1 < q < p$. Then, $-\operatorname{div} a(\nabla u)$ becomes

$$-\Delta_p u - \Delta_q u \quad \text{for all } u \in X,$$

with X being a closed subspace such that $W_0^{1,p}(\Omega) \subseteq X \subseteq W^{1,p}(\Omega)$. In this case the potential has the form $G(\xi) = \frac{1}{p}\|\xi\|^p + \frac{1}{q}\|\xi\|^q$ for all $\xi \in \mathbb{R}^N$. As already mentioned in Sect. 2.3, equations driven by a (p,q)-differential operator arise in mathematical physics such as quantum physics (for existence of soliton solutions, see Benci-D'Avenia-Fortunato-Pisani [BDFP00]) and in plasma physics as well as biophysics (see Cherfils-Il'yasov [CI05]). If $a(\xi) = \left(1+\|\xi\|^2\right)^{\frac{p-2}{2}}\xi$ with $1 < p < \infty$, then this map corresponds to the generalized p-mean curvature differential operator which is given by

$$\operatorname{div}\left[\left(1+\|\nabla u\|^2\right)^{\frac{p-2}{2}}\nabla u\right] \quad \text{for all } u \in X. \tag{2.4.3}$$

Here, the associated potential is given by $G(\xi) = \frac{1}{p}\left[\left(1+\|\xi\|^2\right)^{\frac{p}{2}} - 1\right]$ for all $\xi \in \mathbb{R}^N$. Of course, for $p = 2$, (2.4.3) reduces to the well-known Laplace operator. Another example of a possible operator is given by $a(\xi) = \|\xi\|^{p-2}\xi \pm \frac{\|\xi\|^{p-2}\xi}{1+\|\xi\|^p}$ with $2 \leq p < \infty$ whereby the potential here is $G(\xi) = \frac{1}{p}\|\xi\|^p \pm \frac{1}{p}\ln(1+\|\xi\|^p)$ for all $\xi \in \mathbb{R}^N$.

Even if our operator is possibly nonhomogeneous, it has some nice properties which can be summarized as follows. Let $A\colon X \to X^*$ be the nonlinear map defined by

$$\langle A(u), v\rangle = \int_\Omega a(\nabla u)\cdot\nabla v\,dx \quad \text{for all } u, v \in X, \tag{2.4.4}$$

then A is bounded (that is, it maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone), and fulfills the $(S)_+$-property, that is, if $u_n \rightharpoonup u$ in X and

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \le 0,$$

then $u_n \to u$ in X. Indeed, the $(S)_+$-property is a very powerful tool which under minor assumptions allows the knowledge of strong convergence of weakly convergent sequences.

The motivation for the papers [PW14a**C2**] and [EMPW14**C2**] was the question of if it is possible to give multiplicity results for equations involving nonhomogeneous operators as it was done in the semilinear (Laplacian) and quasilinear (p-Laplacian) cases. Indeed, the lack of homogeneity was the main difficulty and needed some technical refinement in our approach. We will discuss two cases – one with a homogeneous Dirichlet and the other one with a nonlinear Neumann boundary condition.

2.4.1 Homogeneous Dirichlet boundary condition

Consider now a special case of equation (2.4.1) given by

$$-\operatorname{div} a(\nabla u) = \lambda|u|^{q-2}u - f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{2.4.5}$$

with $1 < q \le p$, a parameter $\lambda > 0$, a Carathéodory function $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ being bounded on bounded sets, superlinear at $\pm\infty$, sublinear near zero, and the operator $-\operatorname{div} a(\cdot)$ as introduced in H(a). In addition it satisfies some minor conditions stated on p. 153 given by

$$t \mapsto G_0\left(t^{\frac{1}{q}}\right) \text{ is convex in } (0,+\infty) \text{ and } \limsup_{t\to 0^+}\frac{G_0(t)}{t^q} < +\infty. \tag{2.4.6}$$

The second relation in (2.4.6) implies an estimate like

$$G(\xi) \le \hat{C}\left(\|\xi\|^q + \|\xi\|^p\right) \quad \text{for all } \xi \in \mathbb{R}^N, \tag{2.4.7}$$

with a constant $\hat{C} > 0$ playing an important role which can be seen in the next theorem[9].

Theorem 2.4.1 (Proposition 3.4 on p. 154) *Suppose that hypotheses H(a)' and H(f)$_1$ stated on p. 153 are fulfilled and assume that*

$$\lambda > q\hat{C}\hat{\lambda}_1(q) \quad \text{if } q < p, \qquad \lambda > 2q\hat{C}\hat{\lambda}_1(q) \quad \text{if } q = p$$

with the positive constant $\hat{C}$ in (2.4.7)*. Then problem* (2.4.5) *has at least two nontrivial constant sign solutions*

$$u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad v_0 \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right).$$

[9] Note that the constant $\hat{C}$ is denoted by c_6 in the paper [PW14a**C2**].

By $\hat{\lambda}_1(q) > 0$ we denote the first eigenvalue of the q-Laplacian with a Dirichlet boundary condition. The proof of Theorem 2.4.1 is different from that in Sect. 2.2 concerning the existence of constant sign solutions. Indeed, in Sect. 2.2 we constructed two pairs of sub- and supersolutions based on the scaled first eigenfunction of the p-Laplacian. This time, we used critical point theory to truncated energy functionals. The main challenge was to prove the nontriviality of the critical points which could be realized through the assumptions on the parameter λ.

The further proceeding was to show that problem (2.4.5) has extremal constant sign solutions. This is quite difficult and different to problems with homogeneous operators. The idea was to construct an auxiliary problem based on the assumptions of f. Taking into account the first condition in (2.4.6) we were able to prove the existence of unique constant sign solutions of that auxiliary problem (see Proposition 3.5 on p. 157). Based on this, in combination with Zorn's Lemma, the existence of extremal constant sign solutions to our original problem (2.4.5) could be shown thanks to the directness of the corresponding solution sets (see Proposition 3.6 on p. 159).

The existence of a sign changing solution is now proved within the interval of the extremal constant sign solution. This is realized by applying the mountain-pass theorem, the second deformation theorem, and the variational characterization of the second eigenvalue of the q-Laplacian (see, Cuesta-de Figueiredo-Gossez [CDFG99]). We have the following result.

Theorem 2.4.2 (Proposition 3.7 on p. 161) *If hypotheses H(a)' and H(f)$_1$ stated on p. 153 hold and if*

$$\lambda > q\hat{C}\hat{\lambda}_2(q) \quad \text{if } q < p, \qquad \lambda > 2q\hat{C}\hat{\lambda}_2(q) \quad \text{if } q = p,$$

is satisfied, then problem (2.4.5) *has a nodal solution* $y_0 \in C_0^1\left(\overline{\Omega}\right)$.

We see that we have to strengthen the hypotheses on λ in the case of a nodal solution replacing $\hat{\lambda}_1(q)$ by $\hat{\lambda}_2(q)$. The results presented in [PW14a**C2**] improved former ones for semilinear and quasilinear equations not only due to the nonhomogeneous operator but also to the conditions of the right-hand side nonlinearity. We refer to Remark 3.9 in [PW14a**C2**] for more details. However, we have to mention the work of Carl-Motreanu [CM08] who studied problem (2.4.5) with the same assumptions on f, but with the p-Laplacian instead. Note that if $q = p$ and $a(\xi) = \|\xi\|^{p-2}\xi$ for all $\xi \in \mathbb{R}^N$, then $\hat{C} = \frac{1}{2q}$ and thus $2q\hat{C}\hat{\lambda}_2(q) = \hat{\lambda}_2(q)$. Therefore, Theorems 2.4.1 and 2.4.2 recover the multiplicity results in [CM08].

A second nodal solution to problem (2.4.5) can be obtained by setting $a(\xi) = \|\xi\|^{p-2}\xi + \xi$ for all $\xi \in \mathbb{R}^N$ with $2 < p < \infty$ and $q = 2$ (see Sect. 4 in [PW14a**C2**]). Then, (2.4.5) becomes

$$-\Delta_p u - \Delta u = \lambda u - f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{2.4.8}$$

Moreover, we have to strengthen the hypotheses on f, for example, we now suppose that $f(x,\cdot)$ is differentiable and that the derivative satisfies a subcritical growth condition. The behavior of f at $\pm\infty$ and at zero remains equal, in addition we suppose that $f_s'(x,0) = 0$ uniformly for a.a. $x \in \Omega$.

Under the assumption $\lambda > \hat{\lambda}_2(2)$ where $\hat{\lambda}_2(2)$ denotes the second eigenvalue of the Laplacian and if λ does not belong to the spectrum of the Laplacian, we can prove the existence of four nontrivial solutions to (2.4.8) whereby two of them have constant sign and the others have changing sign, see Theorem 4.2 on p. 166. The proof of this result is a

combination of variational tools and Morse theory, in particular some known properties of critical groups.

2.4.2 Nonhomogeneous, nonlinear Neumann boundary condition

Let us now consider problem (2.4.1) with a nonhomogeneous, nonlinear boundary condition of the form

$$-\operatorname{div} a(\nabla u) = -\chi|u|^{p-2}u - f(x,u) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n_a} = \lambda|u|^{q-2}u - h(x,u) \quad \text{on } \partial\Omega, \tag{2.4.9}$$

where $\frac{\partial u}{\partial n_a} = a(\nabla u)\cdot n$ denotes the generalized normal derivative with $n(x)$ being the outward unit normal at $x \in \partial\Omega$, the mapping $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is as introduced in the beginning of this section, and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$, $h\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are assumed to be Carathéodory functions being bounded on bounded sets, $(p-1)$-superlinear near $\pm\infty$ as well as $(p-1)$-sublinear near zero. In addition, the boundary function shall satisfy a locally Hölder condition near the origin. Here, $1 < q \le p$ and the parameters χ, λ are real, positive numbers such that

$$0 < \chi \begin{cases} < +\infty & \text{if } q < p \\ \le 2p\hat{C} & \text{if } q = p \end{cases}, \qquad \lambda > \begin{cases} q\hat{C}\hat{\lambda}_1(q) & \text{if } q < p \\ 2p\hat{C}\hat{\lambda}_1(p) & \text{if } q = p \end{cases}, \tag{2.4.10}$$

where $\hat{C}$ is the positive constant[10] from estimate (2.4.7) and $\hat{\lambda}_1(r)$ now denotes the first eigenvalue of the Steklov eigenvalue problem of the r-Laplacian given by

$$-\Delta_r u = -|u|^{r-2}u \quad \text{in } \Omega, \qquad \|\nabla u\|^{r-2}\nabla u \cdot n = \hat{\lambda}|u|^{r-2}u \quad \text{on } \partial\Omega, \tag{2.4.11}$$

which was intensively studied by Martínez-Rossi [MR02] (see also Lê [Lê06]). We know that $\hat{\lambda}_1(r)$ is positive, isolated, and simple. Moreover, every eigenfunction corresponding to $\hat{\lambda}_1(r)$ does not change sign in $\overline{\Omega}$, but an eigenfunction associated to an eigenvalue $\lambda > \hat{\lambda}_1(r)$ must change sign in $\overline{\Omega}$.

Under the hypotheses (2.4.10) we could show the existence of two constant sign solutions of (2.4.9), one with positive sign, the other with negative sign (see Proposition 3.3 on p. 182). The idea to deal with such a problem arises from the paper [Win10a] in which the multiplicity of solutions to the equation

$$-\Delta_p u = f(x,u) - |u|^{p-2}u \quad \text{in } \Omega, \qquad \|\nabla u\|^{p-2}\nabla u \cdot n = \lambda|u|^{p-2}u + g(x,u) \quad \text{on } \partial\Omega,$$

has been proved in dependence of the eigenvalues of the Steklov eigenvalue problem stated in (2.4.11). The paper [EMPW14**C2**] extends the results in [Win10a] in different ways. First we see that we could replace the p-Laplacian by a more general possibly nonhomogeneous operator and second, we were able to drop a sign changing condition on f near zero because we used a different approach than in [Win10a]. Actually, we have a third novelty by the appearance of a second parameter in the form of χ. But this was not planned in starting this work, we rather started with $\chi = 1$ and during the proof of Proposition 3.3 on p. 182 we realized that we cannot conclude the nontriviality of our critical points. Therefore, we chose the parameter as given in (2.4.10). Note that if our operator becomes the p-Laplacian we infer $\hat{C} = \frac{1}{2p}$ and so $1 < \chi \le 1$ in the case $p = q$. Hence, the results in [Win10a] are a special case of [EMPW14**C2**].

[10] Note that the constant $\hat{C}$ is denoted by c_7 in the paper [EMPW14**C2**].

The main difference to the results in Subsection 2.4.1 is the appearance of the nonlinear boundary term which made this treatment more complicated. Indeed, the existence of extremal constant sign solutions is again shown by choosing a suitable auxiliary problem. To show the uniqueness of constant sign solutions of that problem, we require further consideration in the form of an integral functional that has a boundary term (see the proof of Proposition 3.4 on p. 186). The exponent in this boundary term is chosen such that after applying the chain rule and Green's identity, the corresponding Gateaux derivatives contain the same boundary terms and therefore vanishes after subtracting (see (3.20) and (3.21) on p. 190). This was one of the main steps.

A sign changing solution of equation (2.4.9) is shown in Proposition 3.6 by strengthen the assumptions on λ meaning that $\hat{\lambda}_1$ in (2.4.10) has to be replaced by $\hat{\lambda}_2$. This is again due to the fact that we used a variational characterization of the second eigenvalue as it was proved in Martínez-Rossi [MR04].

2.4.3 Open problem

In both papers [PW14a**C2**] and [EMPW14**C2**] we observed that in the case of the p-Laplacian the conditions on λ reduce to

$$\lambda > \hat{\lambda}_1(p) \quad \text{and} \quad \lambda > \hat{\lambda}_2(p),$$

with $\hat{\lambda}_1(p), \hat{\lambda}_2(p)$ being the first two eigenvalues of the p-Laplacian with Dirichlet and Steklov boundary condition, respectively. This allows the question of whether there is a relationship between the number $q\widehat{C}\hat{\lambda}_1(q)$ (for $q < p$) and the first eigenvalue of $-\operatorname{div} a(\nabla u)$. First, we have to clarify what is understood by an eigenvalue problem for this operator which is not clear at all. The Dirichlet eigenvalue problem for such an operator could be in the form

$$-\operatorname{div} a(\nabla u) = \lambda |u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{2.4.12}$$

Some authors studied eigenvalue problems like (2.4.12) concerning existence and simplicity of the first eigenvalue, but with the very restrictive property that the operator must be a homogeneous one, see for example Kawohl-Lucia-Prashanth [KLP07] and Lucia-Schuricht [LS13] and the references therein. A more natural generalization, in my opinion, is to define the corresponding eigenvalue problem in the form (recall that $a(\xi) = a_0(\|\xi\|)\xi$ for all $\xi \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$)

$$-\operatorname{div}\left(a_0(\|\nabla u\|)\nabla u\right) = \lambda h(|u|)u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{2.4.13}$$

where $h\colon \mathbb{R} \to \mathbb{R}$ fulfills $h(t) > 0$ for all $t > 0$. So, if we set $a_0(\|\xi\|) = \|\xi\|^{p-2}$ for all $\xi \in \mathbb{R}^N$ and $h(|s|) = |s|^{p-2}$ for all $s \in \mathbb{R}$ it reduces to the eigenvalue problem for the p-Laplacian. Even if it is possible to construct an unbounded nondecreasing sequence (λ_n) of eigenvalues of (2.4.13) via the Ljusternik-Schnirelman theory (see Browder [Bro70] or Zeidler [Zei85]), it is not clear if λ_1 coincides with

$$\lambda^* = \inf_{u \neq 0} \frac{\int_\Omega G(\nabla u)dx}{\int_\Omega H(u)dx}, \tag{2.4.14}$$

where $H_0(s) = \int_0^s sh(t)dt$ and $H(s) = H_0(|s|)$ for all $s \in \mathbb{R}$ due to the lack of homogeneity. We mention a recent work of Montenegro-Lorca [ML12] in which the existence of the first

eigenvalue was proved to nonhomogeneous eigenvalue problems in Orlicz-Sobolev spaces but without the knowledge of its simplicity.

Chapter 3

Multiplicity results for variational-hemivariational inequalities

In 2000, Ricceri presented new abstract critical point results giving the existence of three [Ric00b], resp. infinitely many [Ric00a] critical points of smooth functionals. Such results are very useful – they were the starting point of numerous generalizations of smooth and nonsmooth functionals in recent years. In this chapter we will summarize the main results and ideas of the papers Bonanno-Motreanu-Winkert [BMW11**C3**], Bonanno-Winkert [BW14**C3**], and Bonanno-Motreanu-Winkert [BMW13**C3**] which deals with nonsmooth problems in the sense of variational-hemivariational inequalities as already presented in Sect. 1.3.

Some of the first generalizations of Ricceri's work to nonsmooth problems have been developed by Marano-Motreanu in [MM02a] and [MM02b]. They could be applied to the following inequality problem (see [MM02b]): Find $u \in C$, such that, for all $v \in C$,

$$\int_\Omega \nabla u \cdot (\nabla v - \nabla u) dx + \lambda \int_\Omega \left(J^\circ(x, u; v-u) + (\mu K)^\circ(x, u; v-u)\right) dx \geq 0, \qquad (3.0.1)$$

where C is a closed convex set in $H_0^1(\Omega)$ and $J, K \colon \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions being locally Lipschitz in the second argument with their generalized directional derivatives J°, K° (see Sect. 1.3). Under some assumptions on J, K it is shown that there exists $\delta > 0$ such that to every $\mu \in [-\delta, \delta]$ it corresponds an open interval $\Lambda_\mu \subseteq [0, +\infty[$ for which problem (3.0.1) has at least three solutions whenever $\lambda \in \Lambda_\mu$. The main idea here is to consider an associated nonsmooth functional as the sum of a locally Lipschitz and a convex, proper, and lower semicontinuous functional.

Our focus is on quasilinear inequalities involving the p-Laplacian and with perturbation integrals defined in the domain and/or its boundary. While we suppose in [BMW11**C3**] the continuity of Sobolev functions in $W^{1,p}(\Omega)$ we were able to drop this hypothesis in [BMW13**C3**], [BW14**C3**] giving existence results also in the case $p \leq N$. It should be noted that most results in this direction have been obtained under the very restrictive condition $p > N$. We also refer to the work Motreanu-Winkert [MW10] which deals with quasilinear inequalities having boundary integrals in terms of generalized directional derivatives following the pattern of [MM02a].

3.1 Infinitely many solutions[1]

The main objective in the paper [BMW11**C3**] is the treatment of the following variational inequality: Find $u \in K$ such that, for all $v \in K$,

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(v-u)dx + \int_\Omega q(x)|u|^{p-2}u(v-u)dx \\ + \int_\Omega \lambda\alpha(x)F^\circ(u;v-u)dx + \int_{\partial\Omega}\mu\beta(x)G^\circ(u;v-u)d\sigma \geq 0,\end{aligned} \tag{3.1.1}$$

where K is a closed convex subset of $W^{1,p}(\Omega)$ containing the constant functions, $\alpha \in L^1(\Omega)$, $\beta \in L^1(\partial\Omega)$ with $\alpha(x) \geq 0$ in Ω, $\alpha \not\equiv 0$, $\beta(x) \geq 0$ on $\partial\Omega$, $\beta \not\equiv 0$, and λ, μ are real parameters such that $\lambda > 0$ and $\mu \geq 0$. As before, F° and G° denote Clarke's generalized directional derivatives of locally Lipschitz functions $F, G\colon \mathbb{R} \to \mathbb{R}$ which are defined by

$$F(\xi) = \int_0^\xi f(t)dt, \quad G(\xi) = \int_0^\xi g(t)dt,$$

with given functions $f, g\colon \mathbb{R} \to \mathbb{R}$ being locally essentially bounded. The domain Ω is supposed to be bounded in the Euclidian space $\mathbb{R}^N$, $N \geq 1$, with C^1-boundary $\partial\Omega$ and p belongs to the interval $]N, +\infty[$. Moreover, $q \in L^\infty(\Omega)$ satisfies $q \geq 0, q \not\equiv 0$.

First, we note that since $p > N$ the embedding $W^{1,p}(\Omega) \to C(\overline{\Omega})$ is compact and due to the choice of q it is known that

$$\|u\|_{W^{1,p}(\Omega)} := \left(\int_\Omega (|\nabla u|^p + q(x)|u|^p)dx\right)^{\frac{1}{p}}$$

is an equivalent norm on $W^{1,p}(\Omega)$ (see for instance [Maz85, Section 1.1.15]).

Our aim was to find conditions on the perturbations F and G to prove the existence of infinitely many solutions to inequality (3.1.1). In order to realize this we made use of a critical point result due to Bonanno-Molica Bisci [BMB09] (see Theorem 2.1 in [BMW11**C3**]) presenting different situations of possible critical points to appropriate functionals. Note that this theorem is a version of those in Marano-Motreanu [MM02a, Theorem 1.1].

Based on this result our main theorem stated as Theorem 3.1 on p. 208 gives a precise interval for the parameter λ such that inequality (3.1.1) admits infinitely many solutions provided F satisfies an appropriate oscillating behavior at infinity, G fulfills a suitable growth condition at infinity, and μ is small enough. To be more precise, define the numbers

$$A = \liminf_{\xi\to+\infty} \frac{\max_{|t|\leq\xi}(-F(t))}{\xi^p}, \quad B = \limsup_{\xi\to+\infty} \frac{-F(\xi)}{\xi^p}, \quad \lambda_1 = \frac{\|q\|_1}{\|\alpha\|_1 pB}, \quad \lambda_2 = \frac{1}{\|\alpha\|_1 pc^pA}$$

with c being the embedding constant of $W^{1,p}(\Omega) \to C(\overline{\Omega})$, our main theorem reads as follows.

Theorem 3.1.1 (Theorem 3.1 on p. 208) *Let $\alpha \in L^1(\Omega)$ and $\beta \in L^1(\partial\Omega)$ be nonnegative and nonzero functions. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function and set*

[1] Summary of the paper [BMW11**C3**].

$F(\xi) = \int_0^\xi f(t)\,dt$ *for all* $\xi \in \mathbb{R}$. *Assume that*

$$A < \frac{1}{c^p \|q\|_1} B.$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, *where* λ_1, λ_2 *are given above, for each locally essentially bounded function* $g\colon \mathbb{R} \to \mathbb{R}$, *whose potential* $G(\xi) = \int_0^\xi g(t)\,dt$, $\xi \in \mathbb{R}$ *satisfies*

$$G_\infty := \limsup_{\xi \to +\infty} \frac{\max_{|t| \le \xi}(-G(t))}{\xi^p} < +\infty, \quad \liminf_{\xi \to +\infty}(-G(\xi)) > -\infty,$$

and for every $\mu \in [0, \delta[$, *where*

$$\delta = \delta_{g,\lambda} := \frac{1}{\beta^* p c^p G_\infty} \left(1 - \frac{\lambda}{\lambda_2}\right), \quad (\delta = +\infty \text{ if } G_\infty = 0)\,,$$

with $\beta^* = \int_{\partial\Omega} \beta(x) d\sigma$, *problem* (3.1.1) *admits a sequence of weak solutions that is unbounded in* $W^{1,p}(\Omega)$.

It should be noted that the conditions on F and G can be interchanged, more or less. Indeed, in Theorem 3.1 we supposed an oscillating behavior of F at infinity and a suitable growth on G at infinity. But if we suppose an oscillating behavior on G and a growth condition on F at infinity, we obtain a similar result to Theorem 3.1.1, see Remark 3.5 on p. 211. Moreover, we also give two special versions of Theorem 3.1.1 in the cases that f and g are nonpositive and nonnegative, see Theorems 3.6 and 3.8 on pp. 211 and 212, respectively. Although we need no symmetry conditions on the nonlinear terms we cannot treat the case $p \le N$ since the embedding $W^{1,p}(\Omega) \to C(\overline{\Omega})$ fails. In this case we probably need further assumptions on the nonlinearities.

If F oscillates at zero, we could also give an existence result obtaining a sequence of solutions to (3.1.1) converging uniformly to zero. In fact, defining the values

$$A = \liminf_{\xi \to 0^+} \frac{\max_{|t| \le \xi}(-F(t))}{\xi^p}, \quad B = \limsup_{\xi \to 0^+} \frac{-F(\xi)}{\xi^p}, \quad \lambda_1 = \frac{\|q\|_1}{\|\alpha\|_1 p B}, \quad \lambda_2 = \frac{1}{\|\alpha\|_1 p c^p A},$$

we have the following.

Theorem 3.1.2 (Theorem 3.11 on p. 212) *Let* $\alpha \in L^1(\Omega)$ *and* $\beta \in L^1(\partial\Omega)$ *be nonnegative and non-zero functions. Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a locally essentially bounded function and set* $F(\xi) = \int_0^\xi f(t)\,dt$ *for all* $\xi \in \mathbb{R}$. *Assume that* $A < \frac{1}{c^p \|q\|_1} B$. *Then, for each* $\lambda \in]\lambda_1, \lambda_2[$, *where* λ_1, λ_2 *are given above, for each locally essentially bounded function* $g\colon \mathbb{R} \to \mathbb{R}$, *whose potential* $G(\xi) = \int_0^\xi g(t)\,dt$, $\xi \in \mathbb{R}$ *satisfies*

$$G_0 := \limsup_{\xi \to 0^+} \frac{\max_{|t| \le \xi}(-G(t))}{\xi^p} < +\infty, \quad \liminf_{\xi \to 0^+}(-G(\xi)) \ge 0,$$

and for every $\mu \in [0, \delta[$, *where*

$$\delta = \delta_{g,\lambda} := \frac{1}{\beta^* p c^p G_0} \left(1 - \frac{\lambda}{\lambda_2}\right), \quad (\delta = +\infty \text{ if } G_0 = 0),$$

with $\beta^* = \int_{\partial\Omega} \beta(x) d\sigma$, *problem* (3.1.1) *admits a sequence of distinct weak solutions converging uniformly to zero.*

3.2 An abstract three-solutions-theorem and applications[2]

Now, we are interested in finding the existence of at least three nontrivial solutions to inequalities of type (3.1.1) for both situations $p > N$ and $p \leq N$. Although the model problem in [BW14**C3**] is very similar to those in [BMW11**C3**] (see Sect. 3.1) we state it again due to different notations. As before Ω is a bounded domain with C^1-boundary $\Gamma := \partial\Omega$ and $q \in L^\infty(\Omega)$ fulfills $q \geq 0, q \not\equiv 0$. We consider the following problem: Find $u \in K$ such that, for all $v \in K$,

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla(v-u) dx + \int_\Omega q(x)|u|^{p-2} u(v-u) dx \\ &\quad + \int_\Omega \lambda \alpha(x) F^\circ(u; v-u) dx + \int_\Gamma \mu \beta(x) G^\circ(\gamma u; \gamma v - \gamma u) d\sigma \geq 0, \end{aligned} \tag{3.2.1}$$

where K is a closed convex subset of $W^{1,p}(\Omega), 1 < p < \infty$, now containing the only constant function the zero function (in contrast to Sect. 3.1) while the other data is as it was in the beginning of Sect. 3.1.

The first idea is to develop a new abstract critical point theorem allowing us to apply it to problems like (3.2.1) in the general situation even if $p \leq N$. This result is the following.

Theorem 3.2.1 (Theorem 2.1 on p. 221) *Assume that there is* $r \in]\inf_X \Phi, \sup_X \Phi[$ *such that* $\varphi_1(r) < \varphi_2(r)$. *Further suppose that the functional* J_λ *is bounded from below and satisfies the (PS)-condition for each* $\lambda \in \Lambda := \left]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right[$. *Then, for each* $\lambda \in \Lambda$, J_λ *has three distinct critical points.*

Here, $J_\lambda = \Phi - \lambda\Upsilon + \lambda j$ is a functional defined on a reflexive Banach space X where $\Phi\colon X \to \mathbb{R}$ is sequentially weakly lower semicontinuous as well as coercive, $\Upsilon\colon X \to \mathbb{R}$ is sequentially weakly upper semicontinuous, $j\colon X \to]-\infty, +\infty]$ is supposed to be a convex, proper, and lower semicontinuous functional and φ_1 (resp. φ_2) is defined on p. 221 provided $r > \inf_X \Phi$ (resp. $r < \sup_X \Phi$). Note that J_λ is indeed coercive which follows, for instance, from Corollary 1.3 in Motreanu-Rădulescu [MR03]. The idea in the proof of Theorem 3.2.1 is to show first the existence of local minima $u_1 \in \Phi^{-1}(]-\infty, r[)$ and $u_2 \in \Phi^{-1}(]r, +\infty[)$ of J_λ. Then we may apply a result due to Marano-Motreanu [MM02b, Corollary 2.1] saying that there must be a third critical point of J_λ different from the two local minima.

In addition, we give a variant of Theorem 3.2.1 without supposing that J_λ is bounded from below and fulfills the (PS)-condition. Instead we suppose some other conditions like the convexity of Φ. The precise conditions are stated on p. 223 in (1)–(4).

[2] Summary of the paper [BW14**C3**].

Theorem 3.2.2 (Theorem 2.2 on p. 223) *Assume that there are $r_1, r_2 > 0$ and $\bar{v} \in K$, with $2r_1 < \Phi(\bar{v}) < r_2/2$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Upsilon(u)}{r_1} < \frac{2}{3}\frac{\Upsilon(\bar{v})}{\Phi(\bar{v})} \quad \text{and} \quad \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Upsilon(u)}{r_2} < \frac{1}{3}\frac{\Upsilon(\bar{v})}{\Phi(\bar{v})}.$$

Furthermore, suppose that $\Phi(u) \geq r_2$ for all $u \in \partial K$. Then, for each $\lambda \in \bar{\Lambda}$, where $\bar{\Lambda}$ is given by

$$\bar{\Lambda} := \left]\frac{3}{2}\frac{\Phi(\bar{v})}{\Upsilon(\bar{v})}, \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Upsilon(u)}, \frac{r_2/2}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Upsilon(u)}\right\}\right[,$$

J_λ has three distinct critical points $u_i \in K$ ($i = 1, 2, 3$) such that $\Phi(u_i) < r_2$ for $i \in \{1, 2, 3\}$.

In contrast to Theorem 3.2.1 we have in Theorem 3.2.2 not only the existence of three critical points but also an uniform bound meaning that $u_i \in \Phi^{-1}(]-\infty, r_2[)$. The proof of Theorem 3.2.2 is relying on results of Bonanno-Candito [BC08] and Livrea-Marano [LM04].

The next step is to apply these abstract results to our concrete inequality given in (3.2.1). Let us first say something about the case $p > N$. Recall that c denotes the embedding constant from $W^{1,p}(\Omega)$ into $C(\overline{\Omega})$. Thanks to Theorem 3.2.1 we can prove the subsequent result.

Theorem 3.2.3 (Theorem 3.1 on p. 226) *Let $\alpha \in L^1(\Omega)$, $\beta \in L^1(\Gamma)$ be two non-negative and non-zero functions. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function and put $F(\xi) = \int_0^\xi f(t)dt$ for every $\xi \in \mathbb{R}$. Assume that*

$$\frac{\max_{|t|\leq a_1}(-F(t))}{a_1^p} < \frac{1}{c^p\|q\|_{L^1(\Omega)}}\frac{(-F(a_2))}{a_2^p}, \quad \text{and} \quad \limsup_{|\xi|\to+\infty}\frac{(-F(\xi))}{\xi^p} \leq 0,$$

with positive constants a_1, a_2 satisfying $a_1 < a_2$ and $a_2 \in K$. Then, for each $\lambda \in \Lambda$, where Λ is given by

$$\Lambda := \left]\frac{\|q\|_{L^1(\Omega)}}{p\|\alpha\|_{L^1(\Omega)}}\frac{a_2^p}{(-F(a_2))}, \frac{1}{pc^p\|\alpha\|_{L^1(\Omega)}}\frac{a_1^p}{\max_{|t|\leq a_1}(-F(t))}\right[,$$

and for each locally essentially bounded function $g\colon \mathbb{R} \to \mathbb{R}$, whose potential $G(\xi) = \int_0^\xi g(t)dt$ for every $\xi \in \mathbb{R}$ satisfies

$$\limsup_{|\xi|\to+\infty}\frac{(-G(\xi))}{|\xi|^p} < +\infty,$$

there exists $\overline{\delta} > 0$ such that, for each $\mu \in [0, \overline{\delta}[$, problem (3.2.1) has at least three distinct solutions.

The number $\overline{\delta}$ depends on the given data and is stated on p. 225. That the conditions on F and G are not very restrictive can be seen in Example 3.2 on p. 229. It should also be noted that if K is closed with respect to the maximum and if both f and g are nonpositive, then every solution is nonnegative, see Proposition 3.3 on p. 230. A special version of Theorem

3.2.3 with K being a ball with center zero is then given in Theorem 3.5 on p. 231. Therein we give sufficient conditions for the existence of three nonnegative solutions by applying the abstract results developed in Theorem 3.2.2.

The situation is completely different considering the case $p \leq N$. As already mentioned, the fact that the embedding $W^{1,p}(\Omega) \subset C(\overline{\Omega})$ is now missing brings some difficulties. Instead we will make use of the Sobolev embedding theorem and the properties of the trace mapping. We also assume a subcritical growth on f, that is

$$|f(t)| \leq b_1 + b_2|t|^{s-1}, \tag{3.2.2}$$

for all $t \in \mathbb{R}$ while $s \in \left]1, \frac{Np}{N-p}\right[$ if $p < N$ as well as $s \in]1, +\infty[$ if $p = N$. Let us give our main result in this case.

Theorem 3.2.4 (Theorem 4.1 on p. 234) *Let $\alpha \in L^\infty(\Omega), \beta \in L^\infty(\Gamma)$ be two nonnegative, nonzero functions and let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function satisfying the subcritical growth in (3.2.2). Put $F(\xi) = \int_0^\xi f(t)dt$ for all $\xi \in \mathbb{R}$ and suppose that*

- $K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p} < \frac{p\|\alpha\|_{L^1(\Omega)}}{\|q\|_{L^1(\Omega)}} \frac{[-F(a_2)]}{a_2^p}$ *with positive constants a_1, a_2 satisfying $a_1 < \left(\frac{\|q\|_{L^1(\Omega)}}{p}\right)^{\frac{1}{p}} a_2$ and $a_2 \in K$;*
- $\limsup\limits_{|\xi| \to +\infty} \frac{[-F(\xi)]}{|\xi|^p} \leq 0.$

Then, for each $\lambda \in \widetilde{\Lambda}$, where Λ is given by

$$\widetilde{\Lambda} := \left] \frac{\|q\|_{L^1(\Omega)}}{p\|\alpha\|_{L^1(\Omega)}} \frac{a_2^p}{[-F(a_2)]}, \frac{1}{K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p}} \right[$$

and for each locally essentially bounded function $g\colon \mathbb{R} \to \mathbb{R}$ whose potential $G(\xi) = \int_0^\xi g(t)dt, \xi \in \mathbb{R}$, fulfills

- $[-G(\xi)] \leq b_3|\xi|^p$ *for all $\xi \in \mathbb{R}$ with b_3 being nonnegative,*

there exists $\overline{\delta}$ such that, for each $\mu \in [0, \overline{\delta}[$, problem (3.2.1) *possesses at least three distinct solutions.*

Here, the numbers K_1 and K_2 depend on α, the constants b_1, b_2 from the subcritical growth of f and the embedding constants of the Sobolev embedding $W^{1,p}(\Omega) \to L^s(\Omega)$. One easily verifies that the interval Λ is nonempty. The idea in the proof of Theorem 3.2.4 was again to apply Theorem 3.2.1. In contrast to the case $p > N$ we see that we now have L^∞-functions α, β and the subcritical growth of f is absolutely necessary. All in all the results in [BW14**C3**] present a complete study of nonsmooth problems as in (3.2.1) concerning multiplicity results. Although the starting point is Theorem 3.2.1, the cases $p > N$ and $p \leq N$ need different approaches due to different continuity property of Sobolev functions in $W^{1,p}(\Omega)$.

3.3 Equivalence of constrained and unconstrained problems[3]

The paper [BMW13**C3**] deals with a variational-hemivariational inequality defined for closed convex sets K belonging to the usual Sobolev space $W_0^{1,p}(\Omega)$ consisting of the elements of $W^{1,p}(\Omega)$ having zero traces on the boundary. The novelties of this paper are threefold: First we are interested in developing a multiplicity theorem for such inequalities similar to those presented in Sect. 3.2. The difficulty here is the fact that we cannot suppose that K contains constant functions as it was used in Theorems 3.2.3 and 3.2.4. Second, the results presented here are valid in the more complicated case $p < N$ and third, we give sufficient conditions to prove the equivalence of the constrained and unconstrained problem in case K is the cone of nonnegative elements of $W_0^{1,p}(\Omega)$.

Given a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ and a C^1-boundary $\partial\Omega$ and let $1 < p < N$. We consider the following inequality problem depending on a real parameter $\lambda > 0$: Find $u \in K$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot (\nabla v - \nabla u)dx + \lambda \int_\Omega \alpha(x) F^\circ(u; v-u)dx \geq 0 \quad \text{for all } v \in K, \tag{3.3.1}$$

where $\alpha \in L^1(\Omega)$ such that ess $\inf_{x\in\Omega} \alpha(x) > 0$ and F° stands again for Clarke's generalized directional derivative of a locally Lipschitz function $F\colon \mathbb{R} \to \mathbb{R}$. For the elements of the corresponding gradient in the sense of Clarke, denoted by ∂F, we suppose a subcritical growth, that is

$$|\xi| \leq b_1 + b_2|t|^{s-1} \quad \text{for all } t \in \mathbb{R},\ \xi \in \partial F(t), \tag{3.3.2}$$

with constants $b_1, b_2 \geq 0$ and $1 < s < p^*$. Our existence result can be given as follows.

Theorem 3.3.1 (Theorem 3.1 on p. 245) *Assume that $\alpha \in L^\infty(\Omega)$ fulfills* ess $\inf_{x\in\Omega} \alpha(x) > 0$, *$F\colon \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, with $F(0) = 0$, satisfying the subcritical growth condition (3.3.2) and K is a nonempty, closed, convex subset of $W_0^{1,p}(\Omega)$. In addition, we suppose:*

(H1) $K_1 \dfrac{1}{a_1^{p-1}} + K_2 a_1^{s-p} < \operatorname{ess\,inf}_{x\in\Omega} \alpha(x) \dfrac{[-F(a_2)]}{a_2^p}$ *for positive constants a_1, a_2 with $a_2 > \kappa a_1$ and K_i, $i = 1, 2$, given on p. 245;*

(H2) $\limsup\limits_{|\xi|\to+\infty} \dfrac{-F(\xi)}{|\xi|^p} \leq 0$;

(H3) $-F(t) \geq 0$ *for all $t \in [0, a_2]$;*

(H4) $u_{a_2} \in K$, *where*

$$u_{a_2}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, D), \\ \frac{pa_2}{D}(D - |x - x_0|) & \text{if } x \in B(x_0, D) \setminus B\left(x_0, \frac{(p-1)D}{p}\right), \\ a_2 & \text{if } x \in B\left(x_0, \frac{(p-1)D}{p}\right) \end{cases}$$

[3] Summary of the paper [BMW13**C3**].

for all $x \in \Omega$. Then, for every $\lambda \in \Lambda$ with

$$\Lambda := \left] \frac{p^{p-1}(p^N - (p-1)^N)}{(p-1)^N D^p} \frac{1}{\operatorname{ess\,inf}_{x\in\Omega} \alpha(x)} \frac{a_2^p}{[-F(a_2)]}, \right. \\ \left. \frac{p^{p-1}(p^N - (p-1)^N)}{(p-1)^N D^p} \frac{1}{K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p}} \right[, \tag{3.3.3}$$

problem (3.3.1) *possesses at least three distinct weak solutions.*

Here, K_1, K_2 and κ are suitable chosen constants depending on the data and on the best constant of the Sobolev embedding theorem due to Talenti [Tal76]. We also point out that the interval Λ does not depend on the structure of the set of constraints. Theorem 3.3.1 is proven by using the abstract critical point result stated in Theorem 3.2.1. An interesting closed convex set is the cone of nonnegative elements of $W_0^{1,p}(\Omega)$, that is

$$K_{\geq 0} = \{u \in W_0^{1,p}(\Omega) : u \geq 0 \text{ for a.a. } x \in \Omega\}. \tag{3.3.4}$$

Of course, $K_{\geq 0}$ defined in (3.3.4) fulfills (H4) of Theorem 3.3.1.

A special case of our inequality given in (3.3.1) is the choice $K = W_0^{1,p}(\Omega)$ for which (3.3.1) becomes the unconstrained problem: Find $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + \lambda \int_\Omega \alpha(x) F^\circ(u; v) dx \geq 0 \quad \text{for all } v \in W_0^{1,p}(\Omega). \tag{3.3.5}$$

Recall that inequality (3.3.5) is a so-called hemivariational inequality as already mentioned in Sect. 1.3. Now, we may ask if there is a relationship between the solutions of inequalities (3.3.1) with $K = K_{\geq 0}$ and (3.3.5). Clearly, every nonnegative solution of (3.3.5) is a solution of (3.3.1) with $K_{\geq 0}$ in (3.3.4). An answer about the validity of the converse assertion could be given in the next theorem under a slightly stronger condition as in (H3) of Theorem 3.3.1.

Theorem 3.3.2 (Theorem 4.1 on p. 249) *Assume that $\alpha \in L^\infty(\Omega)$ satisfies* $\operatorname{ess\,inf}_{x\in\Omega} \alpha(x) > 0$ *and $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, with $F(0) = 0$, for which the subcritical growth condition (3.3.2) holds true and in addition*

(H3') there is a constant $a_2 > 0$ such that

$$\xi \leq 0 \quad \text{for all } \xi \in \partial F(t) \text{ with } t \in [0, a_2]. \tag{3.3.6}$$

Then every solution $u \geq 0$ of problem (3.3.1)*, with $K_{\geq 0}$ in* (3.3.4) *and any $\lambda > 0$, that is*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla (v - u) dx + \lambda \int_\Omega \alpha(x) F^\circ(u; v - u) dx \geq 0 \tag{3.3.7}$$

for all $v \in W_0^{1,p}(\Omega)$, $v \geq 0$, and which satisfies $u(x) \in [0, a_2]$ for all $x \in \Omega$, is a (nonnegative) solution of problem (3.3.5)*.*

If $F(0) \leq 0$, then condition (H3') is indeed stronger than (H3) which comes from Lebourg's mean value theorem, see Remark 4.4 on p. 251. Another relationship between our origin problem (3.3.1) and (3.3.5) which concerns a priori estimates is given in the next theorem.

Theorem 3.3.3 (Theorem 4.2 on p. 250) *Assume that* $\alpha \in L^\infty(\Omega)$ *satisfies* $\operatorname{ess\,inf}_{x\in\Omega} \alpha(x) > 0$, $F\colon \mathbb{R} \to \mathbb{R}$ *is a locally Lipschitz function,with* $F(0) = 0$, *for which the subcritical growth condition (3.3.2) holds true, and in addition*

(H5) there is a constant $a_3 > 0$ *such that*

$$\max \partial F(t) \geq 0 \quad \text{for all } t \geq a_3.$$

Then for every solution $u \geq 0$ *of problem* (3.3.1) *with* K *in* (3.3.4), *the following alternatives hold: either* $\inf u < a_3$ *or* u *solves problem* (3.3.5).

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Nonlinear Differential Equations and Applications NoDEA

L^∞-Estimates for nonlinear elliptic Neumann boundary value problems

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Abstract. In this paper we prove the L^∞-boundedness of solutions of the quasilinear elliptic equation

$$\begin{aligned} Au &= f(x,u,\nabla u) && \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} &= g(x,u) && \text{on } \partial\Omega, \end{aligned}$$

where A is a second order quasilinear differential operator and $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ as well as $g : \partial\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions satisfying natural growth conditions. Our main result is given in Theorem 4.1 and is based on the Moser iteration technique along with real interpolation theory. Furthermore, the solutions of the elliptic equation above belong to $C^{1,\alpha}(\overline{\Omega})$, if g is Hölder continuous.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the quasilinear elliptic equation

$$\begin{aligned} Au &= f(x,u,\nabla u) && \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} &= g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\frac{\partial u}{\partial \nu}$ denotes the conormal derivative of u. Here, A is a second-order quasilinear differential operator in divergence form of Leray-Lions type given by

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)), \tag{1.2}$$

and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ as well as $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are some Carathéodory functions. For $u \in W^{1,p}(\Omega)$ defined on the boundary $\partial\Omega$, we make use of the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ which is well known to be compact. For easy readability we will drop the notation γu and write for short u, respectively, $g(x,u) := g(x,\gamma u)$.

The main goal of this paper is to prove a priori estimates for solutions of the nonlinear elliptic equation in (1.1). For this purpose, we use some important tools like the Moser iteration technique and real interpolation theory. By an a priori estimate, we mean an estimate of the form

$$\|u\|_{L^\infty(\Omega)} \leq C,$$

for all possible solutions of (1.1) with some constant C independent of u. Concerning a priori bounds for elliptic equations with zero Neumann conditions we refer to the results in [17] and [19], where they consider problems of the form

$$\begin{aligned} -\Delta u + \lambda u = f(u),\ u > 0 \quad & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad & \text{on } \partial\Omega, \end{aligned}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary and $\lambda > 0$. Motreanu et al. have applied the Moser iteration, too, in [11, Proof of Proposition 2.5] to prove L^∞-boundedness for solutions of the Neumann problem

$$\begin{aligned} -\operatorname{div} \vartheta_\varepsilon(z, \nabla v_\varepsilon) = f_0(z, v_\varepsilon) + \lambda_\varepsilon f_0(z, x_0) - \lambda_\varepsilon |v_\varepsilon - x_0|^{p-2}(v_\varepsilon - x_0) \quad & \text{in } Z, \\ \frac{\partial u}{\partial \nu} = 0 \quad & \text{on } \partial Z, \end{aligned}$$

where

$$\vartheta_\varepsilon(z,\xi) = |\xi|^{p-2}\xi + \lambda_\varepsilon |\nabla x_0|^{p-2}\nabla x_0 + \lambda_\varepsilon |\xi - \nabla x_0|^{p-2}(\xi - \nabla x_0),$$

with $Z \subset \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial Z, 0 < \lambda_\varepsilon \leq 1, \varepsilon \in (0,1]$, $x_0 \in L^\infty(\Omega)$ fixed and with a Carathéodory function $f_0 : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ satisfying some growth condition.

The novelty of our paper is the demonstration of a priori estimates for nonlinear elliptic equations with nonlinear nonhomogenous Neumann boundary values of the form (1.1), where the Carathéodory functions f and g depend on $u, \nabla u$ and u, respectively, satisfying a natural growth condition. Additionally, we extend our results and show that every solution of (1.1) belongs to $C^{1,\alpha}(\overline{\Omega})$ in case g satisfies the condition

$$|g(x_1,s_1) - g(x_2,s_2)| \leq L\left[|x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha\right],$$

for all pairs $(x_1,s_1),(x_2,s_2)$ in $\partial\Omega \times [-M_0, M_0]$, where M_0 is a positive constant and $\alpha \in (0,1]$. The C^1-regularity follows directly from the L^∞-boundedness along with the results of Liebermann in [10].

2. Notations and hypotheses

Let $\frac{1}{p}+\frac{1}{q}=1$. We suppose the following conditions on the operator A and the nonlinearities $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ and $g:\partial\Omega\times\mathbb{R}\to\mathbb{R}$.

(A1) Each $a_i(x,s,\xi)$ satisfies Carathéodory conditions, i.e., is measurable in $x\in\Omega$ for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$ and continuous in (s,ξ) for a.e. $x\in\Omega$. Furthermore, a constant $c_0>0$ and a function $k_0\in L^q(\Omega)$ exist so that

$$|a_i(x,s,\xi)|\le k_0(x)+c_0\left(|s|^{p-1}+|\xi|^{p-1}\right), \tag{2.1}$$

for a.e. $x\in\Omega$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $|\xi|$ denotes the Euclidian norm of the vector ξ.

(A2) The coefficients a_i satisfy a monotonicity condition with respect to ξ in the form

$$\sum_{i=1}^{N}(a_i(x,s,\xi)-a_i(x,s,\xi'))(\xi_i-\xi_i')>0, \tag{2.2}$$

for a.e. $x\in\Omega$, for all $s\in\mathbb{R}$, and for all $\xi,\xi'\in\mathbb{R}^N$ with $\xi\ne\xi'$.

(A3) A constant $c_1>0$ and a function $k_1\in L^\infty(\Omega)$ exist such that

$$\sum_{i=1}^{N}a_i(x,s,\xi)\xi_i\ge c_1|\xi|^p-k_1(x), \tag{2.3}$$

for a.e. $x\in\Omega$, for all $s\in\mathbb{R}$, and for all $\xi\in\mathbb{R}^N$.

(F1) $x\mapsto f(x,s,\xi)$ is measurable in Ω for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$.

(F2) $(s,\xi)\mapsto f(x,s,\xi)$ is continuous in $\mathbb{R}\times\mathbb{R}^N$ for almost all $x\in\Omega$.

(F3) There exists a constant $c_2>0$ such that

$$|f(x,s,\xi)|\le c_2\left(1+|s|^{p-1}+|\xi|^{p-1}\right), \tag{2.4}$$

for a.e. $x\in\Omega$, for all $s\in\mathbb{R}$ and for all $\xi\in\mathbb{R}^N$.

(G1) $x\mapsto g(x,s)$ is measurable in $\partial\Omega$ for all $s\in\mathbb{R}$.

(G2) $s\mapsto g(x,s)$ is continuous in $\mathbb{R}$ for almost all $x\in\partial\Omega$.

(G3) There exists a constant $c_3>0$ such that

$$|g(x,s)|\le c_3(1+|s|^{p-1}), \tag{2.5}$$

for a.e. $x\in\partial\Omega$ and for all $s\in\mathbb{R}$.

Condition (A1) implies that $A:W^{1,p}(\Omega)\to(W^{1,p}(\Omega))^*$ is bounded continuous and along with (A2) it holds that A is pseudomonotone. Due to (A1) the operator A generates a mapping from $W^{1,p}(\Omega)$ into its dual space defined by

$$\langle Au,\varphi\rangle=\int_\Omega\sum_{i=1}^{N}a_i(x,u,\nabla u)\frac{\partial\varphi}{\partial x_i}dx, \tag{2.6}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $W^{1,p}(\Omega)$ and $(W^{1,p}(\Omega))^*$. Assumption (A3) is a coercivity type condition. The conditions (F3) and (G3) ensure that the corresponding Nemytskij operators $F:W^{1,p}(\Omega)\to L^q(\Omega)$ and $G:L^p(\partial\Omega)\to L^q(\partial\Omega)$ defined by

$$Fu(x)=f(x,u(x),\nabla u(x)),\quad Gu(x)=g(x,u(x)), \tag{2.7}$$

are bounded and continuous (see e.g. [18]). The definition of a solution of problem (1.1) in the weak sense is defined as follows.

Definition 2.1. A function $u \in W^{1,p}(\Omega)$ is said to be a weak solution of (1.1) if the following holds

$$\int_\Omega \sum_{i=1}^{N} a_i(x,u,\nabla u)\frac{\partial \varphi}{\partial x_i}dx = \int_\Omega f(x,u,\nabla u)\varphi dx + \int_{\partial\Omega} g(x,u)\varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega).$$

Remark 2.2. The growth conditions on the function f and g can be relaxed, replacing $|s|^{p-1}$ by $|s|^q$ for a suitable $q > p-1$. Thanks to the Sobolev embedding and to the trace embedding, the definition of a weak solution to the Neumann problem would also be consistent in this case. For reasons of simplification, we deal with the given growth conditions as in (2.4) and (2.5).

3. The spaces B^s_{pq} and F^s_{pq}

In this section, we give a brief overview about Besov spaces (respectively, Lizorkin–Triebel spaces) which are needed in the proof of our main theorem.

If $\mathcal{A}$ is a Banach space, then

$$l^\sigma_p(\mathcal{A}) = \left\{ \xi : \xi = \{\xi_j\}_{j=0}^\infty; \xi_j \in \mathcal{A}; \|\xi\|_{l^\sigma_p} = \left(\sum_{j=0}^{\infty} 2^{j\sigma p} \|\xi_j\|^p_{\mathcal{A}} \right)^{\frac{1}{p}} < \infty \right\}$$

for $1 \le p < \infty$ and

$$l^\sigma_\infty(\mathcal{A}) = \left\{ \xi : \xi = \{\xi_j\}_{j=0}^\infty; \xi_j \in \mathcal{A}; \|\xi\|_{l^\sigma_\infty} = \sup_j 2^{j\sigma} \|\xi_j\|_{\mathcal{A}} < \infty \right\}$$

for $p = \infty$ are also Banach spaces (cf. [14, Section 1.18]) where σ is a real number. We recall that $S = S(\mathbb{R}^N)$ is the set of all complex-valued rapidly decreasing infinitely differentiable functions defined on the N-dimensional real Euclidean space $\mathbb{R}^N$. The spaces $S(\mathbb{R}^N)$ and $S'(\mathbb{R}^N)$ (dual space) have their usual topologies, where $S'(\mathbb{R}^N)$ is equipped with the strong topology. We denote by $\mathcal{F}$ the Fourier transform in S and the support of a distribution f is written as $\operatorname{supp} f$. Further, we set

$$M_j = \left\{ \xi : \xi \in \mathbb{R}^N, 2^{j-1} \le |\xi| \le 2^{j+1} \right\}, \quad j = 1, 2, \ldots,$$
$$M_0 = \left\{ \xi : \xi \in \mathbb{R}^N, |\xi| \le 2 \right\}.$$

Then we introduce the spaces $B^s_{pq}(\mathbb{R}^N)$ and $F^s_{pq}(\mathbb{R}^N)$ as follows (see [14, Definition 2.3.1/1]).

Definition 3.1. (a) For $-\infty < s < \infty, 1 < p < \infty$, and $1 \le q < \infty$ one sets

$$B^s_{pq}(\mathbb{R}^N) = \Bigg\{ f : f \in S'(\mathbb{R}^N); f = \sum_{j=0}^{\infty} a_j(x); \operatorname{supp} \mathcal{F}a_j \subset M_j;$$
$$\|\{a_j\}\|_{l^s_q(L_p)} = \left[\sum_{j=0}^{\infty} (2^{sj} \|a_j(x)\|_{L_p})^q \right]^{\frac{1}{q}} < \infty \Bigg\}$$

and for $-\infty < s < \infty, 1 < p < \infty$, and $q = \infty$ one sets

$$B^s_{p\infty}(\mathbb{R}^N) = \Bigg\{ f : f \in S'(\mathbb{R}^N); f = \sum_{j=0}^{\infty} a_j(x); \operatorname{supp} \mathcal{F}a_j \subset M_j;$$
$$\|\{a_j\}\|_{l^s_\infty(L_p)} = \sup_j 2^{sj} \|a_j(x)\|_{L_p} < \infty \Bigg\}.$$

Further, for $-\infty < s < \infty, 1 < p < \infty$ and $1 \le q \le \infty$ let

$$\|f\|_{B^s_{pq}(\mathbb{R}^N)} = \inf_{f = \sum a_j} \|\{a_j\}\|_{l^s_q(L_p)}.$$

(b) For $-\infty < s < \infty, 1 < p < \infty$, and $1 < q < \infty$ one sets

$$F^s_{pq}(\mathbb{R}^N) = \Bigg\{ f : f \in S'(\mathbb{R}^N); f = \sum_{j=0}^{\infty} a_j(x); \operatorname{supp} \mathcal{F}a_j \subset M_j;$$
$$\|\{a_j\}\|_{L_p(l^s_q)} = \left[\int_\Omega \left(\sum_{j=0}^{\infty} 2^{sjq} |a_j(x)|^q \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} < \infty \Bigg\}.$$

Further, let

$$\|f\|_{F^s_{pq}(\mathbb{R}^N)} = \inf_{f = \sum a_j} \|\{a_j\}\|_{L_p(l^s_q)}.$$

(c) For $-\infty < s < \infty$ and $1 < p < \infty$ one sets

$$H^s_p(\mathbb{R}^N) = F^s_{p2}(\mathbb{R}^N).$$

(d) For $1 < p < \infty$ one sets

$$W^s_p(\mathbb{R}^N) = \begin{cases} H^s_p(\mathbb{R}^N) & \text{if } s = 0, 1, 2, \dots, \\ B^s_{pp}(\mathbb{R}^N) & \text{if } 0 < s \neq \text{ integer}. \end{cases}$$

The proof of the following theorem can be found in [14, Theorem 2.3.2].

Theorem 3.2. 1. *Let $-\infty < s < \infty, 1 < p < \infty$, and $1 \le q \le \infty$. Then $B^s_{pq}(\mathbb{R}^N)$ is a Banach space.*

2. *Let $-\infty < s < \infty, 1 < p < \infty$, and $1 < q < \infty$. Then $F^s_{pq}(\mathbb{R}^N)$ is a Banach space.*

It is clear, that all notations above hold true if we replace $\mathbb{R}^N$ by a bounded domain $\Omega \subset \mathbb{R}^N$. The spaces B^s_{pq} and F^s_{pq} are called Besov and Lizorkin–Triebel spaces, respectively, which are equal in case $p = q$ with $1 < p < \infty$ and $-\infty < s < \infty$. We see that the spaces W^s_p with $s = 1, 2, 3, \ldots$ coincide with the well-known Sobolev spaces introduced by S.L.Sobolev and the extension of the definition of the spaces W^s_p to values $0 < s \neq$ integer are the so-called Slobodeckij spaces. Finally, it was shown that H^s_p with $s > 0$ coincide with the well-known Lebesgue (or Liouville, or Bessel-potential) spaces. For more details we refer for example to the books of Triebel in [14–16] or to the monograph of Runst and Sickel in [12].

In our considerations, we need the following continuous embeddings

$$T_1 : B^s_{pp}(\Omega) \to B^{s-\frac{1}{p}}_{pp}(\partial\Omega), \quad \text{with } s > \frac{1}{p},$$
$$T_2 : B^{s-\frac{1}{p}}_{pp}(\partial\Omega) = F^{s-\frac{1}{p}}_{pp}(\partial\Omega) \to F^0_{p2}(\partial\Omega) = L^p(\partial\Omega), \quad \text{with } s > \frac{1}{p},$$

where Ω is a bounded C^∞-domain (see [12, Page 75 and Page 82], [14, 2.3.1 and 2.3.2] and [15, 3.3.1]). Let $s = m + \iota$ with $m \in \mathbb{N}_0$ and $0 \leq \iota < 1$. Then the embeddings are also valid if $\partial\Omega \in C^{m,1}$ [13]. In [3, Satz 9.40] the proof is given for $p = 2$, however, it can be extended to $p \in (1, \infty)$ by using the Fourier transformation in $L^p(\Omega)$ [4].

We set $s = \frac{1}{p} + \widetilde{\varepsilon}$, where $\widetilde{\varepsilon} > 0$ is arbitrarily fixed such that $s = \frac{1}{p} + \widetilde{\varepsilon} < 1$. Thus, the embeddings are valid for a Lipschitz boundary $\partial\Omega$. This yields

$$T_3 : B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega) \to L^p(\partial\Omega),$$

which means

$$\|v\|_{L^p(\partial\Omega)} \leq c_4 \|v\|_{B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega)}, \quad \forall v \in B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega), \tag{3.1}$$

where c_4 is a positive constant. The real interpolation theory implies

$$\big(F^0_{p2}(\Omega), F^1_{p2}(\Omega)\big)_{\frac{1}{p}+\widetilde{\varepsilon},p} = \big(L^p(\Omega), W^{1,p}(\Omega)\big)_{\frac{1}{p}+\widetilde{\varepsilon},p} = B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega),$$

(see [1,14,15], [16, Section 1.6.2 and 1.6.7]) which ensures the norm estimate

$$\|v\|_{B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega)} \leq c_5 \|v\|^{\frac{1}{p}+\widetilde{\varepsilon}}_{W^{1,p}(\Omega)} \|v\|^{1-\frac{1}{p}-\widetilde{\varepsilon}}_{L^p(\Omega)}, \quad \forall v \in W^{1,p}(\Omega), \tag{3.2}$$

(cf. [14, Theorem 1.3.3 (g)]) with a positive constant c_5 only depending on the boundary $\partial\Omega$.

4. Main results

Theorem 4.1. *Let the conditions (A1)–(A3), (F1)–(F3) and (G1)–(G3) be satisfied. Let $u \in W^{1,p}(\Omega)$ be a solution of (1.1). Then $u \in L^\infty(\Omega)$.*

Proof. To prove the $L^\infty-$regularity of u, we will use the Moser iteration technique (see e.g. [5–9]). It suffices to consider the proof in case $1 \leq p \leq N$, otherwise we would be done. First we are going to show that $u^+ = \max\{u, 0\}$

belongs to $L^\infty(\Omega)$. For $M > 0$ we define $v_M(x) = \min\{u^+(x), M\}$. Letting $K(t) = t$ if $t \le M$ and $K(t) = M$ if $t > M$, it follows by [9, Theorem B.3] that $K \circ u^+ = v_M \in W^{1,p}(\Omega)$ and hence $v_M \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. For real $k \ge 0$ we choose $\varphi = v_M^{kp+1}$, then $\nabla\varphi = (kp+1)v_M^{kp}\nabla v_M$ and $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Notice that $u(x) \le 0$ implies directly $v_M(x) = 0$. Testing the weak formulation in Definition 2.1 with $\varphi = v_M^{kp+1}$, one gets

$$\begin{aligned}(kp+1)\int_\Omega \sum_{i=1}^N a_i(x,u^+,\nabla u^+)v_M^{kp}\frac{\partial v_M}{\partial x_i}dx \\ = \int_\Omega f(x,u^+,\nabla u^+)v_M^{kp+1}dx + \int_{\partial\Omega} g(x,u^+)v_M^{kp+1}d\sigma. \end{aligned} \tag{4.1}$$

Applying condition (A3) and the Hölder inequality, the left-hand side of (4.1) can be estimated to obtain

$$\begin{aligned}&(kp+1)\int_\Omega \sum_{i=1}^N a_i(x,u^+,\nabla u^+)v_M^{kp}\frac{\partial v_M}{\partial x_i}dx \\ &= (kp+1)\int_\Omega \sum_{i=1}^N a_i(x,v_M,\nabla v_M)\frac{\partial v_M}{\partial x_i}v_M^{kp}dx \\ &\ge (kp+1)\int_\Omega (c_1|\nabla v_M|^p - k_1)v_M^{kp}dx \\ &\ge c_1\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla v_M^{k+1}|^p dx - e_1(kp+1)\int_\Omega (u^+)^{kp}dx \\ &\ge c_1\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla v_M^{k+1}|^p dx \\ &\quad - e_1(kp+1)|\Omega|^{\frac{1}{k+1}}\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}}, \end{aligned} \tag{4.2}$$

where $e_1 = \|k_1\|_\infty$. The assumption (F3) along with the Hölder inequality and Young's inequality implies

$$\begin{aligned}&\int_\Omega f(x,u^+,\nabla u^+)v_M^{kp+1}dx \\ &\le c_2\int_\Omega (1+|u^+|^{p-1}+|\nabla u^+|^{p-1})v_M^{kp+1}dx \\ &\le c_2|\Omega|^{\frac{p-1}{(k+1)p}}\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} + c_2\int_\Omega (u^+)^{(k+1)p}dx \\ &\quad + c_2\int_\Omega \delta|\nabla u^+|^{(p-1)q}v_M^{k(p-1)q}dx + c_2\int_\Omega C(\delta)v_M^{(k+1)p}dx \\ &\le e_2\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} + (1+C(\delta))c_2\int_\Omega (u^+)^{(k+1)p}dx \\ &\quad + \frac{c_2\delta}{(k+1)^p}\int_\Omega |\nabla(u^+)^{k+1}|^p dx. \end{aligned} \tag{4.3}$$

The same arguments for the boundary integral provide

$$\begin{aligned}\int_{\partial\Omega} g(x,u^+)v_M^{kp+1}d\sigma &\le c_3\int_{\partial\Omega}(1+|u^+|^{p-1})v_M^{kp+1}d\sigma\\ &\le e_3\left(\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}}\\ &\quad + e_4\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma. \end{aligned}\tag{4.4}$$

Applying the estimates (4.2)–(4.4) to (4.1) one gets

$$\begin{aligned}&\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla v_M^{k+1}|^p dx\\ &\le e_2\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} + (1+C(\delta))c_2\int_\Omega (u^+)^{(k+1)p}dx\\ &\quad + \frac{c_2\delta}{(k+1)^p}\int_\Omega |\nabla (u^+)^{k+1}|^p dx + e_3\left(\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}}\\ &\quad + e_4\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma + e_5(kp+1)\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}}. \end{aligned}\tag{4.5}$$

We have $\lim_{M\to\infty} v_M(x) = u^+(x)$ for a.e. $x\in\Omega$ and can apply Fatou's Lemma which results in

$$\begin{aligned}&\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla (u^+)^{k+1}|^p dx\\ &\le e_2\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} + (1+C(\delta))c_2\int_\Omega (u^+)^{(k+1)p}dx\\ &\quad + \frac{c_2\delta}{(k+1)^p}\int_\Omega |\nabla (u^+)^{k+1}|^p dx + e_3\left(\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}}\\ &\quad + e_4\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma + e_5(kp+1)\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}}. \end{aligned}\tag{4.6}$$

We have either

$$\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} \le 1 \quad \text{or} \quad \left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} \le \int_\Omega (u^+)^{(k+1)p}dx,$$

respectively, either

$$\left(\int_{\partial\Omega} (u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}} \le 1 \quad \text{or} \quad \left(\int_{\partial\Omega} (u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}} \le \int_{\partial\Omega} (u^+)^{(k+1)p}d\sigma,$$

and finally, either

$$\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}} \le 1 \quad \text{or} \quad \left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}} \le \int_\Omega (u^+)^{(k+1)p}dx.$$

Using the calculation above to (4.6), we obtain

$$\left[\frac{kp+1}{(k+1)^p} - \frac{c_2\delta}{(k+1)^p}\right]\int_\Omega |\nabla(u^+)^{k+1}|^p dx$$
$$\leq (C(\delta)c_2 + e_6(kp+1))\int_\Omega (u^+)^{(k+1)p}dx + e_7\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma + e_8, \quad (4.7)$$

where the choice $\delta = \frac{kp+1}{2c_2}$ results in

$$\frac{kp+1}{2(k+1)^p}\int_\Omega |\nabla(u^+)^{k+1}|^p dx$$
$$\leq e_9(kp+1)\int_\Omega (u^+)^{(k+1)p}dx + e_7\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma + e_8. \quad (4.8)$$

It should be pointed out that

$$C(\delta) = (\delta p)^{-\frac{q}{p}}\cdot\frac{1}{q} = \left(\frac{2c_2}{p}\right)^{\frac{q}{p}}\cdot\left(\frac{1}{kp+1}\right)^{\frac{q}{p}}\cdot\frac{1}{q} \leq e_{10}$$

with a positive constant e_{10} where we have set $e_9 = e_{10}c_2 + e_6$. Adding on both sides of (4.8) the positive integral $\frac{kp+1}{2(k+1)^p}\int_\Omega (u^+)^{(k+1)p}dx$ yields

$$\frac{kp+1}{2(k+1)^p}\left[\int_\Omega |\nabla(u^+)^{k+1}|^p dx + \int_\Omega (u^+)^{(k+1)p}dx\right]$$
$$\leq e_{11}(kp+1)\int_\Omega (u^+)^{(k+1)p}dx + e_7\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma + e_8, \quad (4.9)$$

due to the fact that $\frac{kp+1}{2(k+1)^p} < kp+1$ for all $k \geq 0$. Next we want to estimate the boundary integral by an integral in the domain Ω. Using (3.1), (3.2) and Young's inequality yields

$$\int_{\partial\Omega}\left((u^+)^{k+1}\right)^p d\sigma$$
$$= \|(u^+)^{k+1}\|^p_{L^p(\partial\Omega)}$$
$$\leq c_4^p\|(u^+)^{k+1}\|^p_{B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega)}$$
$$\leq c_4^p c_5^p\|(u^+)^{k+1}\|^{\left(\frac{1}{p}+\widetilde{\varepsilon}\right)p}_{W^{1,p}(\Omega)}\|(u^+)^{k+1}\|^{\left(1-\frac{1}{p}-\widetilde{\varepsilon}\right)p}_{L^p(\Omega)}$$
$$\leq c_4^p c_5^p\left(\delta'\|(u^+)^{k+1}\|^{(1+\widetilde{\varepsilon}p)\widetilde{q}}_{W^{1,p}(\Omega)} + C(\delta')\|(u^+)^{k+1}\|^{(p-1-\widetilde{\varepsilon}p)\widetilde{q}'}_{L^p(\Omega)}\right)$$
$$= c_4^p c_5^p\left(\delta'\|(u^+)^{k+1}\|^p_{W^{1,p}(\Omega)} + C(\delta')\|(u^+)^{k+1}\|^p_{L^p(\Omega)}\right), \quad (4.10)$$

where $\widetilde{q} = \frac{p}{1+\widetilde{\varepsilon}p}$ and $\widetilde{q}' = \frac{p}{p-1-\widetilde{\varepsilon}p}$ satisfy $\frac{1}{\widetilde{q}} + \frac{1}{\widetilde{q}'} = 1$ and δ' is a free parameter specified later. Note that the positive constant $C(\delta')$ depends on δ'. Applying

(4.10) to (4.9) shows

$$\begin{aligned}
&\frac{kp+1}{2(k+1)^p}\left[\int_\Omega |\nabla (u^+)^{k+1}|^p dx + \int_\Omega (u^+)^{(k+1)p} dx\right] \\
&\quad \le e_{11}(kp+1)\int_\Omega (u^+)^{(k+1)p} dx + e_7 \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma + e_8 \\
&\quad \le e_{11}(kp+1)\int_\Omega (u^+)^{(k+1)p} dx + e_{12}\delta' \|(u^+)^{k+1}\|^p_{W^{1,p}(\Omega)} \\
&\qquad + e_{12} C(\delta') \|(u^+)^{k+1}\|^p_{L^p(\Omega)} + e_8,
\end{aligned}$$

where $e_{12} = e_7 c_4^p c_5^p$ is a positive constant. We take $\delta' = \frac{kp+1}{e_{12} 4(k+1)^p}$ to get

$$\begin{aligned}
&\left(\frac{kp+1}{2(k+1)^p} - e_{12}\frac{kp+1}{e_{12}4(k+1)^p}\right)\left[\int_\Omega |\nabla (u^+)^{k+1}|^p dx + \int_\Omega |(u^+)^{k+1}|^p dx\right] \\
&\quad \le e_{11}(kp+1)\int_\Omega (u^+)^{(k+1)p} dx + e_{12} C(\delta') \|(u^+)^{k+1}\|^p_{L^p(\Omega)} + e_8 \\
&\quad \le e_{13}(kp+1+C(\delta'))\int_\Omega (u^+)^{(k+1)p} dx + e_8, \qquad (4.11)
\end{aligned}$$

where it holds

$$\begin{aligned}
kp+1+C(\delta') &= kp+1+\left(\frac{4e_{12}}{p}\right)^{\frac{q}{p}} \cdot \left(\frac{(k+1)^p}{kp+1}\right)^{\frac{q}{p}} \cdot \frac{1}{q} \\
&\le e_{14}\left(kp+1+\left(\frac{(k+1)^{\frac{p}{p-1}}}{(kp+1)^{\frac{1}{p-1}}}\right)\right) \\
&\le e_{15}(kp+1)^{\frac{p}{p-1}}.
\end{aligned}$$

Applying the calculations above to (4.11) provides

$$\begin{aligned}
&\frac{kp+1}{4(k+1)^p}\left[\int_\Omega |\nabla (u^+)^{k+1}|^p dx + \int_\Omega |(u^+)^{k+1}|^p dx\right] \\
&\quad \le e_{16}(kp+1)^{\frac{p}{p-1}}\left[\int_\Omega (u^+)^{(k+1)p} dx + 1\right],
\end{aligned}$$

equivalently

$$\|(u^+)^{k+1}\|^p_{W^{1,p}(\Omega)} \le (kp+1)^{\frac{1}{p-1}}(k+1)^p e_{17}\left[\int_\Omega (u^+)^{(k+1)p} dx + 1\right].$$

By Sobolev's embedding theorem a positive constant e_{18} exists such that

$$\|(u^+)^{k+1}\|_{L^{p^*}(\Omega)} \le e_{18}\|(u^+)^{k+1}\|_{W^{1,p}(\Omega)}, \qquad (4.12)$$

where $p^* = \frac{Np}{N-p}$ if $1 < p < N$ and $p^* = 2p$ if $p = N$. We get

$$\begin{aligned}
&\|u^+\|_{L^{(k+1)p^*}(\Omega)} \\
&\quad = \|(u^+)^{k+1}\|^{\frac{1}{k+1}}_{L^{p^*}(\Omega)} \\
&\quad \le e_{18}^{\frac{1}{k+1}} \|(u^+)^{k+1}\|^{\frac{1}{k+1}}_{W^{1,p}(\Omega)} \\
&\quad \le e_{18}^{\frac{1}{k+1}} \left((kp+1)^{\frac{1}{(p-1)p}}(k+1)\right)^{\frac{1}{k+1}} e_{17}^{\frac{1}{(k+1)p}} \left[\int_\Omega (u^+)^{(k+1)p})dx + 1\right]^{\frac{1}{(k+1)p}}.
\end{aligned}$$

Since $((kp+1)^{\frac{1}{(p-1)p}}(k+1))^{\frac{1}{\sqrt{k+1}}} \ge 1$ and $\lim_{k\to\infty}((kp+1)^{\frac{1}{(p-1)p}}(k+1))^{\frac{1}{\sqrt{k+1}}} = 1$, there exists a constant $e_{19} > 1$ such that $((kp+1)^{\frac{1}{(p-1)p}}(k+1))^{\frac{1}{k+1}} \le e_{19}^{\frac{1}{\sqrt{k+1}}}$. We obtain

$$\|u^+\|_{L^{(k+1)p^*}(\Omega)} \le e_{18}^{\frac{1}{k+1}} e_{19}^{\frac{1}{\sqrt{k+1}}} e_{17}^{\frac{1}{(k+1)p}} \left[\int_\Omega (u^+)^{(k+1)p})dx + 1\right]^{\frac{1}{(k+1)p}}. \tag{4.13}$$

Now, we will use the bootstrap arguments similarly as in the proof of [8, Lemma 3.2] starting with $(k_1+1)p = p^*$ to get

$$\|u^+\|_{L^{(k+1)p^*}(\Omega)} \le c(k)$$

for any finite number $k > 0$ which shows that $u^+ \in L^r(\Omega)$ for any $r \in (1,\infty)$. To prove the uniform estimate with respect to k we argue as follows. If there is a sequence $k_n \to \infty$ such that

$$\int_\Omega (u^+)^{(k_n+1)p} dx \le 1,$$

we have immediately

$$\|u^+\|_{L^\infty(\Omega)} \le 1,$$

(cf. the proof of [8, Lemma 3.2]). In the opposite case there exists $k_0 > 0$ such that

$$\int_\Omega (u^+)^{(k+1)p} dx > 1$$

for any $k \ge k_0$. Then we conclude from (4.13)

$$\|u^+\|_{L^{(k+1)p^*}(\Omega)} \le e_{18}^{\frac{1}{k+1}} e_{19}^{\frac{1}{\sqrt{k+1}}} e_{20}^{\frac{1}{(k+1)p}} \|u^+\|_{L^{(k+1)p}}, \quad \text{for any } k \ge k_0, \tag{4.14}$$

where $e_{20} = 2e_{17}$. Choosing $k := k_1$ such that $(k_1+1)p = (k_0+1)p^*$ yields

$$\|u^+\|_{L^{(k_1+1)p^*}(\Omega)} \le e_{18}^{\frac{1}{k_1+1}} e_{19}^{\frac{1}{\sqrt{k_1+1}}} e_{20}^{\frac{1}{(k_1+1)p}} \|u^+\|_{L^{(k_1+1)p}(\Omega)}. \tag{4.15}$$

Next, we can choose k_2 in (4.14) such that $(k_2+1)p = (k_1+1)p^*$ to get

$$\begin{aligned}
\|u^+\|_{L^{(k_2+1)p^*}(\Omega)} &\le e_{18}^{\frac{1}{k_2+1}} e_{19}^{\frac{1}{\sqrt{k_2+1}}} e_{20}^{\frac{1}{(k_2+1)p}} \|u^+\|_{L^{(k_2+1)p}(\Omega)} \\
&= e_{18}^{\frac{1}{k_2+1}} e_{19}^{\frac{1}{\sqrt{k_2+1}}} e_{20}^{\frac{1}{(k_2+1)p}} \|u^+\|_{L^{(k_1+1)p^*}(\Omega)}.
\end{aligned} \tag{4.16}$$

By induction we obtain

$$\begin{aligned}\|u^+\|_{L^{(k_n+1)p^*}(\Omega)} &\leq e_{18}^{\frac{1}{k_n+1}} e_{19}^{\frac{1}{\sqrt{k_n+1}}} e_{20}^{\frac{1}{(k_n+1)p}} \|u^+\|_{L^{(k_n+1)p}(\Omega)} \\ &= e_{18}^{\frac{1}{k_n+1}} e_{19}^{\frac{1}{\sqrt{k_n+1}}} e_{20}^{\frac{1}{(k_n+1)p}} \|u^+\|_{L^{(k_{n-1}+1)p^*}(\Omega)}, \end{aligned} \tag{4.17}$$

where the sequence (k_n) is chosen such that $(k_n+1)p = (k_{n-1}+1)p^*$ with $k_0 > 0$. One easily verifies that $k_n + 1 = (\frac{p^*}{p})^n$. Thus

$$\begin{aligned}&\|u^+\|_{L^{(k_n+1)p^*}(\Omega)} \\ &= e_{18}^{\sum_{i=1}^n \frac{1}{k_i+1}} e_{19}^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} e_{20}^{\sum_{i=1}^n \frac{1}{(k_i+1)p}} \|u^+\|_{L^{(k_0+1)p^*}(\Omega)}, \end{aligned} \tag{4.18}$$

with $r_n = (k_n+1)p^* \to \infty$ as $n \to \infty$. Since $\frac{1}{k_i+1} = (\frac{p}{p^*})^i$ and $\frac{p}{p^*} < 1$ there is a constant $e_{21} > 0$ such that

$$\|u^+\|_{L^{(k_n+1)p^*}(\Omega)} \leq e_{21}\|u^+\|_{L^{(k_0+1)p^*}(\Omega)} < \infty. \tag{4.19}$$

Let us assume that $u^+ \notin L^\infty(\Omega)$. Then there exist $\eta > 0$ and a set A of positive measure in Ω such that $u^+(x) \geq e_{21}\|u^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta$ for $x \in A$. It follows that

$$\begin{aligned}\|u^+\|_{L^{(k_n+1)p^*}(\Omega)} &\geq \left(\int_A |u^+(x)|^{(k_n+1)p^*}\right)^{\frac{1}{(k_n+1)p^*}} \\ &\geq \left(e_{21}\|u^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta\right) |A|^{\frac{1}{(k_n+1)p^*}}. \end{aligned}$$

Passing to the limes inferior in the inequality above yields

$$\liminf_{n\to\infty} \|u^+\|_{L^{(k_n+1)p^*}(\Omega)} \geq e_{21}\|u^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta,$$

which is a contradiction to (4.19) and hence, $u^+ \in L^\infty(\Omega)$. In a similar way one shows that $u^- = \max\{-u, 0\} \in L^\infty(\Omega)$. This proves $u = u^+ - u^- \in L^\infty(\Omega)$. □

Remark 4.2. Note that the finiteness of the integrals

$$\int_\Omega |\nabla(u^+)^{k+1}|^p dx, \quad \int_\Omega |(u^+)^{k+1}|^p dx$$

is shown in the end of the proof of Theorem 4.1 by a suitable choice of the parameter k. This is a typical proceeding in the use of the Moser iteration (see, e.g. [8]).

Let us now suppose an additional condition to the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ as follows.

(G4) g satisfies the condition

$$|g(x_1, s_1) - g(x_2, s_2)| \leq L\left[|x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha\right],$$

for all pairs $(x_1, s_1), (x_2, s_2)$ in $\partial\Omega \times [-M_0, M_0]$, where M_0 is a positive constant and $\alpha \in (0, 1]$.

Theorem 4.3. *Let the conditions (A1)–(A3), (F1)–(F3) and (G1)–(G4) be satisfied. Let $u \in W^{1,p}(\Omega)$ be a solution of (1.1). Then $u \in C^{1,\alpha}(\overline{\Omega})$.*

Proof. Theorem 4.1 implies $u \in L^\infty(\Omega)$. Moreover, we see at once that the assumptions (0.3a)–(0.3d) and (0.6) in [10] are satisfied which yields in view of [10, Theorem 2] the assertion. □

Example. Let $A = -\Delta_p, 1 < p < \infty$, be the negative $p-$Laplacian which is defined by

$$-\Delta_p u = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{where } \nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N). \tag{4.20}$$

The coefficients $a_i, i = 1, \ldots, N$ are given by

$$a_i(x, s, \xi) = |\xi|^{p-2}\xi_i.$$

Thus, hypothesis (A1) is satisfied with $k_0 = 0$ and $c_0 = 1$. Hypothesis (A2) is a consequence of the inequalities from the vector-valued function $\xi \mapsto |\xi|^{p-2}\xi$ (see [2, Page 37]) and (A3) is satisfied with $c_1 = 1$ and $k_1 = 0$. Our equation in (1.1) gets the form

$$\begin{aligned} -\Delta_p u &= f(x, u, \nabla u) \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= g(x, u) \quad \text{on } \partial\Omega, \end{aligned} \tag{4.21}$$

where $\frac{\partial u}{\partial \nu}$ means the outer normal derivative of u with respect to $\partial\Omega$. Theorems 4.1 and 4.3 ensure under the assumptions (F1)–(F3) and (G1)–(G4) that every solution u of (4.21) satisfies $u \in L^\infty(\Omega)$ and $u \in C^{1,\alpha}(\overline{\Omega})$.

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A PRIORI BOUNDS FOR WEAK SOLUTIONS TO ELLIPTIC EQUATIONS WITH NONSTANDARD GROWTH

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Abstract. In this paper we study elliptic equations with a nonlinear conormal derivative boundary condition involving nonstandard growth terms. By means of the localization method and De Giorgi's iteration technique we derive global a priori bounds for weak solutions of such problems.

1. **Introduction.** The present paper is concerned with global a priori bounds for elliptic equations with nonlinear conormal derivative boundary conditions which may contain nonlinearities with variable growth exponents. More precisely, let Ω be a bounded domain in $\mathbb{R}^N, N > 1$, with Lipschitz boundary $\Gamma := \partial\Omega$ and let $p \in C(\overline{\Omega})$ be a function that satisfies $1 < p^- := \inf_{\overline{\Omega}} p(x)$. We deal with elliptic equations of the form

$$\begin{aligned} -\operatorname{div} \mathcal{A}(x,u,\nabla u) &= \mathcal{B}(x,u,\nabla u) && \text{in } \Omega, \\ \mathcal{A}(x,u,\nabla u)\cdot \nu &= \mathcal{C}(x,u) && \text{on } \Gamma, \end{aligned} \tag{1.1}$$

where $\nu(x)$ denotes the outer unit normal of Ω at $x \in \Gamma$, and $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ satisfy suitable $p(x)$-structure conditions, see (H) below.

An important special case of (1.1) which fits in our setting is given by

$$-\Delta_{p(x)} u = \mathcal{B}(x,u,\nabla u) \quad \text{in } \Omega, \qquad |\nabla u|^{p(x)-2}\partial_\nu u = \mathcal{C}(x,u) \quad \text{on } \Gamma.$$

Here the operator $\operatorname{div} \mathcal{A}$ becomes the so-called $p(x)$-Laplacian

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u),$$

which reduces to the standard p-Laplacian if $p(x) \equiv p$.

In recent years there has been a growing interest in the study of elliptic problems with a $p(x)$-structure, which are also termed problems with nonstandard growth conditions. Equations of this type appear in the study of non-Newtonian fluids with thermo-convective effects (see Antontsev and Rodrigues [4], Zhikov [35]), electro-rheological fluids (see Diening [8], Rajagopal and Růžička [29], Růžička [31]), the

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thermistor problem (see Zhikov [36]), or the problem of image recovery (see Chen et al. [5]).

Throughout the paper we impose the following conditions.

(H) The functions $\mathcal{A}:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$, $\mathcal{B}:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$, and $\mathcal{C}:\Gamma\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions satisfying the subsequent structure conditions:

$$\text{(H1)}\quad |\mathcal{A}(x,s,\xi)|\le a_0|\xi|^{p(x)-1}+a_1|s|^{q_0(x)\frac{p(x)-1}{p(x)}}+a_2, \qquad \text{for a.a. } x\in\Omega,$$

$$\text{(H2)}\quad \mathcal{A}(x,s,\xi)\cdot\xi\ge a_3|\xi|^{p(x)}-a_4|s|^{q_0(x)}-a_5, \qquad \text{for a.a. } x\in\Omega,$$

$$\text{(H3)}\quad |\mathcal{B}(x,s,\xi)|\le b_0|\xi|^{p(x)\frac{q_0(x)-1}{q_0(x)}}+b_1|s|^{q_0(x)-1}+b_2, \qquad \text{for a.a. } x\in\Omega,$$

$$\text{(H4)}\quad |\mathcal{C}(x,s)|\le c_0|s|^{q_1(x)-1}+c_1, \qquad \text{for a.a. } x\in\Gamma,$$

and for all $s\in\mathbb{R}$, and all $\xi\in\mathbb{R}^N$. Here a_i,b_j and c_l are positive constants, $p\in C(\overline{\Omega})$ with $\inf_{\overline{\Omega}}p(x)>1$, and $q_0\in C(\overline{\Omega})$ as well as $q_1\in C(\Gamma)$ are chosen such that

$$p(x)\le q_0(x)<p^*(x),\quad x\in\overline{\Omega},\quad\text{and}\quad p(x)\le q_1(x)<p_*(x),\quad x\in\Gamma,$$

with the critical exponents

$$p^*(x)=\begin{cases}\frac{Np(x)}{N-p(x)} & \text{if } p(x)<N,\\ +\infty & \text{if } p(x)\ge N,\end{cases}\qquad p_*(x)=\begin{cases}\frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x)<N,\\ +\infty & \text{if } p(x)\ge N.\end{cases}$$

A function $u\in W^{1,p(\cdot)}(\Omega)$ is said to be a **weak solution** (**subsolution, supersolution**) of equation (1.1) if

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi dx = (\le,\ge)\int_\Omega \mathcal{B}(x,u,\nabla u)\varphi dx+\int_\Gamma \mathcal{C}(x,u)\varphi d\sigma, \tag{1.2}$$

holds for all nonnegative test functions $\varphi\in W^{1,p(\cdot)}(\Omega)$, where $d\sigma$ denotes the usual $(N-1)$-dimensional surface measure.

This definition makes sense, since thanks to assumption (H) the integrals in (1.2) are finite, by Hölder's inequality and embedding results for $W^{1,p(\cdot)}(\Omega)$-functions, see below.

The main goal of this paper is to prove a priori bounds for weak sub- and supersolutions, in particular for weak solutions of problem (1.1). Using the notation $y_+=\max(y,0)$, our main result reads as follows.

Theorem 1.1. *Let the assumptions in (H) be satisfied. Then there exist positive constants $\alpha=\alpha(p,q_0,q_1)$ and $C=C(p,q_0,q_1,a_3,a_4,a_5,b_0,b_1,b_2,c_0,c_1,N,\Omega)$ such that the following assertions hold.*

(i) If $u\in W^{1,p(\cdot)}(\Omega)$ is a weak subsolution of (1.1) then

$$\operatorname*{ess\,sup}_\Omega u\le 2\max\left(1,C\left[\int_\Omega u_+^{q_0(x)}dx+\int_\Gamma u_+^{q_1(x)}d\sigma\right]^\alpha\right).$$

(ii) If $u\in W^{1,p(\cdot)}(\Omega)$ is a weak supersolution of (1.1) then

$$\operatorname*{ess\,inf}_\Omega u\ge -2\max\left(1,C\left[\int_\Omega (-u)_+^{q_0(x)}dx+\int_\Gamma (-u)_+^{q_1(x)}d\sigma\right]^\alpha\right).$$

Note that the constants a_0,a_1,a_2, which appear in (H1), do not play any role in determining the constants α and C. The finiteness of the right-hand sides in (i) and (ii) is a consequence of the compact embedding $W^{1,p(\cdot)}(\Omega)\to L^{q_0(\cdot)}(\Omega)$ and the fact that the trace operator is a bounded operator from $W^{1,p(\cdot)}(\Omega)$ into $L^{q_1(\cdot)}(\Gamma)$ (see

Fan et al. [15, Theorem 1.3] and Fan [12, Corollary 2.4]). We further point out that we merely assume continuity for the variable exponents p, q_0, and q_1; log-Hölder continuity conditions are not required.

Our proof of Theorem 1.1 uses De Giorgi's iteration technique and the localization method. By means of the latter we are able to reduce the estimates involving variable exponents to ones with constant exponents, which then also allows us to apply classical embedding results. This crucial step in the proof is achieved by means of an appropriate partition of unity.

By the definition of sub- and supersolution of (1.1) one easily verifies that a weak solution is both a weak subsolution and a weak supersolution. Hence we have the following.

Corollary 1.2. *Let the assumptions (H) be satisfied and let $u \in W^{1,p(\cdot)}(\Omega)$ be a weak solution of (1.1). Then $u \in L^\infty(\Omega)$ and the estimates in (i) and (ii) from Theorem 1.1 are valid.*

The main novelty of the paper consists in the generality of the assumptions needed to establish the boundedness of weak solutions to (1.1). In particular the assumptions on the nonlinearity $\mathcal{C}$ are rather general, allowing for a growth term with variable exponent, which seems to be optimal. Another novelty is the use of the localization technique in the context of global a priori estimates for problems with variable exponents and nonlinear conormal derivative boundary conditions.

Let us comment on some relevant known results on elliptic problems with $p(x)$-structure. Local boundedness of solutions to the equation

$$-\operatorname{div} \mathcal{A}(x, u, \nabla u) = \mathcal{B}(x, u, \nabla u) \quad \text{in } \Omega, \tag{1.3}$$

has been studied by Fan and Zhao [16]. There it is shown that under suitable structure conditions every weak solution u of (1.3) (corresponding to test functions $\varphi \in W_0^{1,p(x)}(\Omega)$) belongs to $L^\infty_{\mathrm{loc}}(\Omega)$, and if in addition u is bounded on the boundary Γ, then $u \in L^\infty(\Omega)$. The proof uses De Giorgi iterations as well. Recently, Gasiński and Papageorgiou (see [19, Proposition 3.1]) studied global a priori bounds for weak solutions to the equation

$$\begin{aligned} -\Delta_{p(x)} u &= g(x,u) && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \Gamma, \end{aligned} \tag{1.4}$$

where the Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies a subcritical growth condition. They proved that every weak solution $u \in W^{1,p(\cdot)}(\Omega)$ of problem (1.4) belongs to $L^\infty(\Omega)$ provided $p \in C^1(\overline{\Omega})$ satisfying $1 < \min_{x \in \overline{\Omega}} p(x)$.

L^∞-estimates for solutions of (1.1) in case $p(x) \equiv p$ with $q_0(x) = q_1(x) \equiv p$ have been established by the first author in [33, 34] following Moser's iteration technique (for constant p see also Pucci and Servadei [28]).

Concerning boundedness and regularity results for problems of type (1.3), in particular for the special case

$$-\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = 0,$$

we further refer to Acerbi and Mingione [1, 2], Antontsev and Consiglieri [3], Chiadò Piat and Coscia [6], Diening et al. [9], Eleuteri and Habermann [11], Fan [13, 14], Fan and Zhao [18], Habermann and Zatorska-Goldstein [20], Harjulehto et al. [21], Liskevich and Skrypnik [25], Lukkari [26, 27] and the references given therein.

The paper is organized as follows. In Section 2 we fix some notation and recall the definition of the variable exponent spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. We further state a lemma on sequences of numbers which will be needed for the De Giorgi iterations. The main result is proved in Section 3. From the structure conditions we first derive truncated energy estimates. These are then used, together with the localization method and embedding results, to prove suitable iterative inequalities, which in turn imply the desired a priori bounds.

2. **Notations and preliminaries.** Suppose that Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary Γ and let $p \in C(\overline{\Omega})$ with $p(x) > 1$ for all $x \in \overline{\Omega}$. We set $p^- := \min_{x\in\overline{\Omega}} p(x)$ and $p^+ := \max_{x\in\overline{\Omega}} p(x)$, then $p^- > 1$ and $p^+ < \infty$. By $L^{p(\cdot)}(\Omega)$ we identify the variable exponent Lebesgue space which is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u \;\middle|\; u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)} dx < +\infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \tau > 0 : \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \le 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{p(\cdot)}(\Omega)}.$$

For more information and basic properties of variable exponent spaces we refer the reader to the papers of Fan and Zhao [17], Kováčik and Rákosník [22] and the recent monograph of Diening et al. [10]. If $p(x) \equiv p$ is a constant, the usual Sobolev space $W^{1,p}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx \right)^{\frac{1}{p}}.$$

For $q_0 \in C(\overline{\Omega})$ and $q_1 \in C(\Gamma)$ (as in (H)) we define

$$q_0^+ = \max_{\overline{\Omega}} q_0(x), \quad q_0^- = \min_{\overline{\Omega}} q_0(x),$$
$$q_1^+ = \max_{\Gamma} q_1(x), \quad q_1^- = \min_{\Gamma} q_1(x).$$

For $s \in [1, \infty)$ we further use the notation

$$s^* = \begin{cases} \frac{Ns}{N-s} & \text{if } s < N, \\ +\infty & \text{if } s \ge N, \end{cases} \qquad s_* = \begin{cases} \frac{(N-1)s}{N-s} & \text{if } s < N, \\ +\infty & \text{if } s \ge N. \end{cases}$$

The following lemma concerning the geometric convergence of sequences of numbers will be needed for the De Giorgi iteration arguments below. It can be found, for example in [32]. The case $\delta_1 = \delta_2$ is contained in [23, Chapter II, Lemma 5.6], see also [7, Chapter I, Lemma 4.1].

Lemma 2.1. *Let $\{Y_n\}, n = 0, 1, 2, \ldots$, be a sequence of positive numbers, satisfying the recursion inequality*

$$Y_{n+1} \le K b^n \left(Y_n^{1+\delta_1} + Y_n^{1+\delta_2} \right), \quad n = 0, 1, 2, \ldots,$$

for some $b > 1$, $K > 0$, *and* $\delta_2 \geq \delta_1 > 0$. *If*

$$Y_0 \leq (2K)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}},$$

then

$$Y_n \leq (2K)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}} b^{-\frac{n}{\delta_1}}, \quad n \in \mathbb{N},$$

in particular $\{Y_n\} \to 0$ *as* $n \to \infty$.

3. **Truncated energy estimates and proof of Theorem 1.1.** Our proof of the sup-bounds for weak subsolutions of (1.1) is based on the following lemma on truncated energy estimates.

Lemma 3.1. *Let the conditions in (H) be satisfied. If* u *is a weak subsolution of (1.1), there holds*

$$\int_{A_k} |\nabla u|^{p(x)} dx \leq d_1 \int_{A_k} u^{q_0(x)} dx + d_2 \int_{\Gamma_k} u^{q_1(x)} d\sigma,$$

where

$$A_k = \{x \in \Omega : u(x) > k\}, \qquad \Gamma_k = \{x \in \Gamma : u(x) > k\}, \quad k \geq 1,$$

and $d_1 = 2a_3^{-1}(a_4+a_5+b_1+b_2+b_0\varepsilon^{-(q_0^+ +1)})$, $d_2 = 2a_3^{-1}(c_0+c_1)$, *and* $\varepsilon = \min(1, \frac{a_3}{2b_0})$.

Proof. Let $u \in W^{1,p(\cdot)}(\Omega)$ be a weak subsolution of (1.1) and let $k \geq 1$. Taking $\varphi = (u-k)_+ = \max(u-k, 0) \in W^{1,p(\cdot)}(\Omega)$ (see [24, Lemma 3.2]) as test function in (1.2) with the '$\leq$'-sign we obtain

$$\begin{aligned} &\int_{A_k} \mathcal{A}(x, u, \nabla u) \cdot \nabla(u-k) dx \\ &\leq \int_{A_k} \mathcal{B}(x, u, \nabla u)(u-k) dx + \int_{\Gamma_k} \mathcal{C}(x, u)(u-k) d\sigma. \end{aligned} \tag{3.1}$$

Using the structure condition (H2) we estimate the left-hand side of (3.1) as follows.

$$\begin{aligned} &\int_{A_k} \mathcal{A}(x, u, \nabla u) \cdot \nabla(u-k) dx \\ &= \int_{A_k} \mathcal{A}(x, u, \nabla u) \cdot \nabla u dx \\ &\geq \int_{A_k} \left(a_3 |\nabla u|^{p(x)} - a_4 |u|^{q_0(x)} - a_5 \right) dx \\ &\geq a_3 \int_{A_k} |\nabla u|^{p(x)} dx - (a_4 + a_5) \int_{A_k} |u|^{q_0(x)} dx, \end{aligned} \tag{3.2}$$

as $u^{q_0(x)} > u > 1$ in A_k. Now, we are going to estimate the right-hand side of (3.1). By Young's inequality with $\varepsilon \in (0,1]$ and condition (H3) we have

$$\begin{aligned}
&\int_{A_k} \mathcal{B}(x,u,\nabla u)(u-k)dx \\
&\leq \int_{A_k} \left[b_0 |\nabla u|^{p(x)\frac{q_0(x)-1}{q_0(x)}} + b_1 |u|^{q_0(x)-1} + b_2 \right] (u-k)dx \\
&\leq b_0 \int_{A_k} \left[\varepsilon^{\frac{q_0(x)-1}{q_0(x)}} |\nabla u|^{p(x)\frac{q_0(x)-1}{q_0(x)}} \varepsilon^{-\frac{q_0(x)-1}{q_0(x)}} u \right] dx + (b_1+b_2) \int_{A_k} |u|^{q_0(x)} dx \qquad (3.3) \\
&\leq b_0 \int_{A_k} \varepsilon |\nabla u|^{p(x)} dx + b_0 \int_{A_k} \varepsilon^{-(q_0(x)-1)} u^{q_0(x)} dx + (b_1+b_2) \int_{A_k} |u|^{q_0(x)} dx \\
&\leq \varepsilon b_0 \int_{A_k} |\nabla u|^{p(x)} dx + \left(b_0 \varepsilon^{-(q_0^+-1)} + b_1 + b_2 \right) \int_{A_k} u^{q_0(x)} dx.
\end{aligned}$$

Thanks to condition (H4), the boundary integral can be estimated through

$$\begin{aligned}
\int_{\Gamma_k} \mathcal{C}(x,u)(u-k)d\sigma &\leq \int_{\Gamma_k} (c_0 |u|^{q_1(x)-1} + c_1)(u-k)d\sigma \\
&\leq (c_0+c_1) \int_{\Gamma_k} u^{q_1(x)} d\sigma,
\end{aligned} \qquad (3.4)$$

as $u > 1$ on Γ_k. Combining (3.1)–(3.4) and choosing $\varepsilon = \min(1, \frac{a_3}{2b_0})$ gives

$$\begin{aligned}
&\frac{a_3}{2} \int_{A_k} |\nabla u|^{p(x)} dx \\
&\leq \left(a_4 + a_5 + b_1 + b_2 + b_0 \varepsilon^{-(q_0^+ + 1)} \right) \int_{A_k} u^{q_0(x)} dx + (c_0+c_1) \int_{\Gamma_k} u^{q_1(x)} d\sigma.
\end{aligned}$$

Dividing the last inequality by $\frac{a_3}{2} > 0$ yields the assertion of the lemma. □

The corresponding result for supersolutions reads as follows.

Lemma 3.2. *Let the conditions in (H) be satisfied. If u is a weak supersolution of (1.1), there holds*

$$\int_{\tilde{A}_k} |\nabla u|^{p(x)} dx \leq d_1 \int_{\tilde{A}_k} (-u)^{q_0(x)} dx + d_2 \int_{\tilde{\Gamma}_k} (-u)^{q_1(x)} d\sigma,$$

where

$$\tilde{A}_k = \{x \in \Omega : -u(x) > k\}, \qquad \tilde{\Gamma}_k = \{x \in \Gamma : -u(x) > k\}, \quad k \geq 1,$$

and d_1 and d_2 are the same constants as in Lemma 3.1.

Proof. The proof is analogous to the previous one. We take $\varphi = -(u+k)_- = -\min(u+k, 0) \geq 0$ as test function in (1.2), which now holds with the '$\geq$'-sign, and use the same arguments as in the proof of Lemma 3.1. This yields the asserted inequality. □

Now we are in position to prove the main result of this paper.

Proof of Theorem 1.1.

(i) Definition of the iteration variables Z_n, $\tilde{Z}_n$, and basic estimates. Let now

$$k_n = k\left(2 - \frac{1}{2^n}\right), \quad n = 0, 1, 2, \ldots,$$

with $k \geq 1$ specified later and put

$$Z_n := \int_{A_{k_n}} (u - k_n)^{q_0(x)} dx, \qquad \tilde{Z}_n := \int_{\Gamma_{k_n}} (u - k_n)^{q_1(x)} d\sigma.$$

We have

$$\begin{aligned} Z_n &\geq \int_{A_{k_{n+1}}} (u - k_n)^{q_0(x)} dx \geq \int_{A_{k_{n+1}}} u^{q_0(x)} \left(1 - \frac{k_n}{k_{n+1}}\right)^{q_0(x)} dx \\ &\geq \int_{A_{k_{n+1}}} \frac{1}{2^{q_0(x)(n+2)}} u^{q_0(x)} dx, \end{aligned}$$

and thus

$$\int_{A_{k_{n+1}}} u^{q_0(x)} dx \leq 2^{q_0^+(n+2)} Z_n. \tag{3.5}$$

Analogously, we see that

$$\int_{\Gamma_{k_{n+1}}} u^{q_1(x)} d\sigma \leq 2^{q_1^+(n+2)} \tilde{Z}_n. \tag{3.6}$$

From (3.5)–(3.6) and Lemma 3.1 with k being replaced by $k_{n+1} \geq 1$ it follows that

$$\int_{A_{k_{n+1}}} |\nabla (u - k_{n+1})|^{p(x)} dx \leq d_3 a^n (Z_n + \tilde{Z}_n), \tag{3.7}$$

where $d_3 = \max\left(d_1 2^{2q_0^+}, d_2 2^{2q_1^+}\right)$ and $a = \max\left(2^{q_0^+}, 2^{q_1^+}\right)$.

Furthermore, we have

$$\begin{aligned} |A_{k_{n+1}}| &\leq \int_{A_{k_{n+1}}} \left(\frac{u - k_n}{k_{n+1} - k_n}\right)^{q_0(x)} dx \\ &\leq \int_{A_{k_n}} \frac{2^{q_0(x)(n+1)}}{k^{q_0(x)}} (u - k_n)^{q_0(x)} dx \\ &\leq \frac{2^{q_0^+(n+1)}}{k^{q_0^-}} \int_{A_{k_n}} (u - k_n)^{q_0(x)} dx \\ &= \frac{2^{q_0^+(n+1)}}{k^{q_0^-}} Z_n. \end{aligned} \tag{3.8}$$

(ii) Partition of unity. By compactness of $\overline{\Omega}$, for any $R > 0$ there exists a finite open cover $\{B_i(R)\}_{i=1,\dots,m}$ of balls $B_i := B_i(R)$ with radius R such that $\overline{\Omega} \subset \bigcup_{i=1}^m B_i(R)$. Moreover, since $p \in C(\overline{\Omega})$, $q_0 \in C(\overline{\Omega})$, and $q_1 \in C(\Gamma)$, these functions are uniformly continuous on $\overline{\Omega}$ and Γ, respectively. Recalling that

$$p(x) \leq q_0(x) < p^*(x), \quad x \in \overline{\Omega}, \quad \text{and} \quad p(x) \leq q_1(x) < p_*(x), \quad x \in \Gamma,$$

we may take $R > 0$ small enough such that

$$p_i^+ \leq q_{0,i}^+ < (p_i^-)^*, \qquad p_i^+ \leq q_{1,i}^+ < (p_i^-)_*, \qquad i = 1, \dots, m,$$

where

$$\begin{aligned} p_i^+ &= \max_{B_i \cap \overline{\Omega}} p(x), \quad q_{0,i}^+ = \max_{B_i \cap \overline{\Omega}} q_0(x), \\ p_i^- &= \min_{B_i \cap \overline{\Omega}} p(x), \quad q_{1,i}^+ = \max_{B_i \cap \Gamma} q_1(x). \end{aligned}$$

We next choose a partition of unity $\{\xi_i\}_{i=1}^m \subset C_0^\infty(\mathbb{R}^N)$ associated to the open cover $\{B_i(R)\}_{i=1,\dots,m}$ (see e.g. [30, Thm. 6.20]), that is, we have

$$\operatorname{supp}\xi_i \subset B_i, \quad 0 \le \xi_i \le 1, \ \ i = 1, \dots, m, \quad \text{and} \ \ \sum_{i=1}^m \xi_i = 1 \ \text{ on } \overline{\Omega}.$$

Let $L > 0$ be a positive constant such that

$$|\nabla \xi_i| \le L, \quad i = 1, \dots, m.$$

(iii) Estimating the gradient term in (3.7) from below. Using the partition of unity from step (ii) we have

$$\begin{aligned}
&\int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p(x)} dx \\
&= \int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p(x)} \sum_{i=1}^m \xi_i dx \\
&\ge \sum_{i=1}^m \int_{A_{k_{n+1}}} (|\nabla(u - k_{n+1})|^{p_i^-} - 1)\xi_i dx \\
&\ge \sum_{i=1}^m \int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p_i^-} \xi_i^{p_i^-} dx - m|A_{k_{n+1}}|,
\end{aligned} \tag{3.9}$$

as $\xi_i \ge \xi_i^{p_i^-}$. From (3.9) we trivially deduce that for all $i = 1, \dots, m$,

$$\int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p(x)} dx \ge \int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p_i^-} \xi_i^{p_i^-} dx - m|A_{k_{n+1}}|. \tag{3.10}$$

Combining (3.7) and (3.10) and using (3.8) yields

$$\int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p_i^-} \xi_i^{p_i^-} dx \le d_4 a^n (Z_n + \tilde{Z}_n) \tag{3.11}$$

for any $i = 1, \dots, m$, with the positive constant $d_4 = d_3 + m2^{q_0^+}$.

(iv) Estimating Z_{n+1}. Next we want to derive a suitable estimate for the term Z_{n+1} from above. To this end, we make again use of the partition of unity

introduced in step (ii). We have

$$\begin{aligned} Z_{n+1} &= \int_{A_{k_{n+1}}} (u-k_{n+1})^{q_0(x)}dx \\ &= \int_{A_{k_{n+1}}} (u-k_{n+1})^{q_0(x)} \left(\sum_{i=1}^m \xi_i\right)^{q_0^+} dx \\ &\le \int_{A_{k_{n+1}}} (u-k_{n+1})^{q_0(x)} m^{q_0^+} \sum_{i=1}^m \xi_i^{q_0^+} dx \\ &\le m^{q_0^+} \sum_{i=1}^m \int_{A_{k_{n+1}}} (u-k_{n+1})^{q_0(x)} \xi_i^{q_0(x)} dx \\ &\le m^{q_0^+} \sum_{i=1}^m \left[\int_{A_{k_{n+1}}} (u-k_{n+1})^{q_{0,i}^+} \xi_i^{q_{0,i}^+} dx + \int_{A_{k_{n+1}}} (u-k_{n+1})^{q_{0,i}^-} \xi_i^{q_{0,i}^-} dx\right], \end{aligned} \tag{3.12}$$

where we have set $q_{0,i}^- = \min_{B_i \cap \overline{\Omega}} q_0(x)$. Observe that $p_i^- \le q_{0,i}^- \le q_{0,i}^+ < (p_i^-)^*$ for all $i = 1, \dots, m$.

Let now $i \in \{1, \dots, m\}$ be fixed, and suppose that $r \in \{q_{0,i}^-, q_{0,i}^+\}$. Then $p_i^- \le r < (p_i^-)^*$ and $r \le q^+$, where $q^+ = \max(q_0^+, q_1^+)$. By Hölder's inequality and the continuous embedding $W^{1,p_i^-}(\Omega) \hookrightarrow L^{(p_i^-)^*}(\Omega)$, we may estimate as follows.

$$\begin{aligned} &\int_\Omega (u-k_{n+1})_+^r \xi_i^r dx \\ &\le \left(\int_\Omega (u-k_{n+1})_+^{(p_i^-)^*} \xi_i^{(p_i^-)^*} dx\right)^{\frac{r}{(p_i^-)^*}} |A_{k_{n+1}}|^{1-\frac{r}{(p_i^-)^*}} \\ &\le C^{q^+} \left(\int_\Omega \left[|\nabla[(u-k_{n+1})_+\xi_i]|^{p_i^-} \right.\right. \\ &\qquad \left.\left. + |(u-k_{n+1})_+ \xi_i|^{p_i^-}\right] dx\right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{1-\frac{r}{(p_i^-)^*}}, \end{aligned} \tag{3.13}$$

where $C = \max(1, C(p_1^-, N), \dots, C(p_m^-, N))$ with $C(p_j^-, N)$ being the embedding constant corresponding to the embedding $W^{1,p_j^-}(\Omega) \hookrightarrow L^{(p_j^-)^*}(\Omega)$, $j = 1, \dots, m$. Thus C is independent of i. A simple calculation shows that the right-hand side of (3.13) can be estimated further to obtain

$$\begin{aligned} &\int_\Omega (u-k_{n+1})_+^r \xi_i^r dx \\ &\le d_5 \left(\int_\Omega |\nabla (u-k_{n+1})_+|^{p_i^-} \xi_i^{p_i^-} dx\right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{1-\frac{r}{(p_i^-)^*}} \\ &\quad + d_6 \left(\int_{A_{k_{n+1}}} u^{q_0(x)} dx\right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{1-\frac{r}{(p_i^-)^*}}. \end{aligned} \tag{3.14}$$

Here $d_5 = d_5(q^+, C)$ and $d_6 = d_6(p^+, q^+, C, L)$ are positive constants, where L is the constant introduced in step (ii).

Using (3.5), (3.11), (3.12), and (3.14) with $r = q_{0,i}^+$ and $r = q_{0,i}^-$, respectively, for $i = 1, \dots, m$, we get

$$\begin{aligned} Z_{n+1} \le m^{q_0^+} \sum_{i=1}^m \Bigg[& d_5 \left(d_4 a^n (Z_n + \tilde{Z}_n) \right)^{\frac{q_{0,i}^+}{p_i^-}} |A_{k_{n+1}}|^{1 - \frac{q_{0,i}^+}{(p_i^-)^*}} \\ & + d_6 \left(2^{q_0^+(n+2)} Z_n \right)^{\frac{q_{0,i}^+}{p_i^-}} |A_{k_{n+1}}|^{1 - \frac{q_{0,i}^+}{(p_i^-)^*}} \\ & + d_5 \left(d_4 a^n (Z_n + \tilde{Z}_n) dx \right)^{\frac{q_{0,i}^-}{p_i^-}} |A_{k_{n+1}}|^{1 - \frac{q_{0,i}^-}{(p_i^-)^*}} \\ & + d_6 \left(2^{q_0^+(n+2)} Z_n \right)^{\frac{q_{0,i}^-}{p_i^-}} |A_{k_{n+1}}|^{1 - \frac{q_{0,i}^-}{(p_i^-)^*}} \Bigg]. \end{aligned} \tag{3.15}$$

Setting

$$Y_n := Z_n + \tilde{Z}_n$$

and $\eta = \max \left(\frac{q_{0,1}^+}{(p_1^-)^*}, \dots, \frac{q_{0,m}^+}{(p_m^-)^*} \right)$ we have for $r \in \{q_{0,i}^+, q_{0,i}^-\}$

$$\begin{aligned} \left(d_4 a^n (Z_n + \tilde{Z}_n) \right)^{\frac{r}{p_i^-}} &\le (d_4)^{\frac{q^+}{p^-}} \left(a^{\frac{q^+}{p^-}} \right)^n (Y_n + Y_n^{\frac{q^+}{p^-}}), \\ \left(2^{q_0^+(n+2)} Z_n \right)^{\frac{r}{p_i^-}} &\le \left(2^{\frac{(q^+)^2(n+2)}{p^-}} \right) (Y_n + Y_n^{\frac{q^+}{p^-}}), \\ |A_{k_{n+1}}|^{1 - \frac{r}{(p_i^-)^*}} &\le 2^{q^+(n+1)} \left(\frac{1}{k^{q_0^-}} \right)^{1-\eta} (Y_n + Y_n^{1-\eta}). \end{aligned} \tag{3.16}$$

Using these estimates, we conclude from (3.15) that

$$Z_{n+1} \le d_7 d_8^n \frac{1}{k^{q_0^-(1-\eta)}} \left(Y_n^2 + Y_n^{2-\eta} + Y_n^{1+\frac{q^+}{p^-}} + Y_n^{1+\frac{q^+}{p^-}-\eta} \right) \tag{3.17}$$

where d_7 and d_8 are positive constants that only depend on the data.

(v) Estimating $\tilde{Z}_{n+1}$. We proceed similarly as in step (iv). Analogously to (3.12) we get an estimate for $\tilde{Z}_{n+1}$ of the form

$$\tilde{Z}_{n+1} \le m^{q_1^+} \sum_{i=1}^m \left[\int_{\Gamma_{k_{n+1}}} (u - k_{n+1})^{q_{1,i}^+} \xi_i^{q_{1,i}^+} d\sigma + \int_{\Gamma_{k_{n+1}}} (u - k_{n+1})^{q_{1,i}^-} \xi_i^{q_{1,i}^-} d\sigma \right], \tag{3.18}$$

where $q_{1,i}^- = \min_{B_i \cap \Gamma} q_1(x)$. Note that $p_i^- \le q_{1,i}^- \le q_{1,i}^+ < (p_i^-)_*$ for $i = 1, \dots, m$.

Let now $i \in \{1, \dots, m\}$ be fixed, and suppose that $r \in \{q_{1,i}^-, q_{1,i}^+\}$, that is we have $p_i^- \le r < (p_i^-)_*$ and $r \le q^+$. Define $s = s_i(r) \in (1, N)$ by means of

$$s_* = \begin{cases} \frac{r + (p_i^-)_*}{2} & \text{if } (p_i^-)_* < \infty, \\ r + 1 & \text{if } (p_i^-)_* = \infty. \end{cases}$$

Then $s < p_i^- \le r < s_* < (p_i^-)_*$. Since the trace operator maps $W^{1,s}(\Omega)$ boundedly into $L^r(\Gamma)$, we have, using Hölder's inequality,

$$\begin{aligned}
&\int_\Gamma ((u-k_{n+1})_+\xi_i)^r d\sigma \\
&\le \hat{C}^r \left[\int_\Omega \left(|\nabla[(u-k_{n+1})_+\xi_i]|^s + |(u-k_{n+1})_+\xi_i|^s\right) dx\right]^{\frac{r}{s}} \\
&\le \hat{C}^{q^+} \left[\int_\Omega \left(|\nabla[(u-k_{n+1})_+\xi_i]|^{p_i^-} + |(u-k_{n+1})_+\xi_i|^{p_i^-}\right) dx\right]^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s}{p_i^-}\right)\frac{r}{s}},
\end{aligned}$$

where $\hat{C}$ denotes the maximum of 1 and the norms of the trace maps $\gamma : W^{1,s}(\Omega) \to L^r(\Gamma)$ when r runs through set $\bigcup_{j=1}^m \{q_{1,j}^-, q_{1,j}^+\}$. The right-hand side of the last inequality can be estimated to get

$$\begin{aligned}
&\int_\Gamma ((u-k_{n+1})_+\xi_i)^r d\sigma \\
&\le d_9 \left(\int_\Omega |\nabla(u-k_{n+1})_+|^{p_i^-} \xi_i^{p_i^-} dx\right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s}{p_i^-}\right)\frac{r}{s}} \\
&\quad + d_{10} \left(\int_{A_{k_{n+1}}} u^{q_0(x)} dx\right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s}{p_i^-}\right)\frac{r}{s}}
\end{aligned} \tag{3.19}$$

with positive constants d_9, d_{10} that only depend on the data.

From (3.5), (3.11), (3.18), and (3.19) with $r = q_{1,i}^+$ and $r = q_{1,i}^-$, respectively, for $i = 1, \dots, m$, we infer that

$$\begin{aligned}
\tilde{Z}_{n+1} \le m^{q_1^+} \sum_{i=1}^m \Bigg[& d_9 \left(d_4 a^n (Z_n + \tilde{Z}_n)\right)^{\frac{q_{1,i}^+}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s_i(q_{1,i}^+)}{p_i^-}\right)\frac{q_{1,i}^+}{s_i(q_{1,i}^+)}} \\
& + d_{10} \left(2^{q_0^+(n+2)} Z_n\right)^{\frac{q_{1,i}^+}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s_i(q_{1,i}^+)}{p_i^-}\right)\frac{q_{1,i}^+}{s_i(q_{1,i}^+)}} \\
& + d_9 \left(d_4 a^n (Z_n + \tilde{Z}_n)\right)^{\frac{q_{1,i}^-}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s_i(q_{1,i}^-)}{p_i^-}\right)\frac{q_{1,i}^-}{s_i(q_{1,i}^-)}} \\
& + d_{10} \left(2^{q_0^+(n+2)} Z_n\right)^{\frac{q_{1,i}^-}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s_i(q_{1,i}^-)}{p_i^-}\right)\frac{q_{1,i}^-}{s_i(q_{1,i}^-)}} \Bigg].
\end{aligned} \tag{3.20}$$

Put $\tilde{\eta} = \max\left(\frac{s_1(q_{1,1}^+)}{p_1^-}, \dots, \frac{s_m(q_{1,m}^+)}{p_m^-}\right)$. Similarly to (3.16) we have for $r \in \{q_{1,i}^+, q_{1,i}^-\}$

$$\begin{aligned}
&\left(d_4 a^n (Z_n + \tilde{Z}_n)\right)^{\frac{r}{p_i^-}} \le (d_4)^{\frac{q^+}{p^-}} \left(a^{\frac{q^+}{p^-}}\right)^n (Y_n + Y_n^{\frac{q^+}{p^-}}), \\
&\left(2^{q_0^+(n+2)} Z_n\right)^{\frac{r}{p_i^-}} \le \left(2^{\frac{(q^+)^2(n+2)}{p^-}}\right) (Y_n + Y_n^{\frac{q^+}{p^-}}), \\
&|A_{k_{n+1}}|^{\left(1-\frac{s_i(r)}{p_i^-}\right)\frac{r}{s_i(r)}} \le 2^{q^+(n+1)} \left(\frac{1}{k^{q_0^-}}\right)^{1-\tilde{\eta}} (Y_n^{q^+} + Y_n^{1-\tilde{\eta}}).
\end{aligned} \tag{3.21}$$

Finally, (3.20) and (3.21) imply that

$$\tilde{Z}_{n+1} \leq d_{11} d_{12}^n \frac{1}{k^{q_0^-(1-\tilde{\eta})}} \left(Y_n^{q^+ 1} + Y_n^{2-\tilde{\eta}} + Y_n^{q^+ + \frac{q^+}{p^-}} + Y_n^{1+\frac{q^+}{p^-}-\tilde{\eta}} \right) \tag{3.22}$$

where d_{11} and d_{12} are positive constants that only depend on the data.

(vi) The iterative inequality for Y_n. Recall that $Y_n = Z_n + \tilde{Z}_n$. Hence (3.17) and (3.22) yield

$$\begin{aligned} Y_{n+1} &\leq K b^n \frac{1}{k^{q_0^-(1-\hat{\eta})}} \left(Y_n^2 + Y_n^{2-\eta} + Y_n^{1+\frac{q^+}{p^-}} + Y_n^{1+\frac{q^+}{p^-}-\eta} \right. \\ &\qquad \left. + Y_n^{q^+} + Y_n^{2-\tilde{\eta}} + Y_n^{q^+ + \frac{q^+}{p^-}} + Y_n^{1+\frac{q^+}{p^-}-\tilde{\eta}} \right) \\ &\leq 8 K b^n \frac{1}{k^{q_0^-(1-\hat{\eta})}} \left(Y_n^{1+\delta_1} + Y_n^{1+\delta_2} \right) \end{aligned}$$

with $K = \max(d_7, d_{11}), b = \max(d_8, d_{12})$, $\hat{\eta} = \max(\eta, \tilde{\eta})$, and where $0 < \delta_1 \leq \delta_2$ are given by

$$\begin{aligned} \delta_1 &= \min\left(1, 1-\eta, \frac{q^+}{p^-}, \frac{q^+}{p^-} - \eta, q^+, 1-\tilde{\eta}, q^+ + \frac{q^+}{p^-} - 1, \frac{q^+}{p^-} - \tilde{\eta}\right), \\ \delta_2 &= \max\left(1, 1-\eta, \frac{q^+}{p^-}, \frac{q^+}{p^-} - \eta, q^+, 1-\tilde{\eta}, q^+ + \frac{q^+}{p^-} - 1, \frac{q^+}{p^-} - \tilde{\eta}\right). \end{aligned}$$

Without loss of generality we may assume that $b > 1$. Now we may apply Lemma 2.1, which says that $Y_n \to 0$ as $n \to \infty$ provided

$$Y_0 = \int_\Omega (u-k)_+^{q_0(x)} dx + \int_\Gamma (u-k)_+^{q_1(x)} d\sigma \leq \left(\frac{16K}{k^{q_0^-(1-\hat{\eta})}} \right)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}}. \tag{3.23}$$

Relation (3.23) is clearly satisfied if

$$\int_\Omega u_+^{q_0(x)} dx + \int_\Gamma u_+^{q_1(x)} d\sigma \int_\Gamma \leq \left(\frac{16K}{k^{q_0^-(1-\hat{\eta})}} \right)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}}. \tag{3.24}$$

Hence, if we choose k such that

$$k = \max\left(1, \left[(16K)^{\frac{1}{\delta_1}} b^{\frac{1}{\delta_1^2}} \left(\int_\Omega u_+^{q_0(x)} dx + \int_\Gamma u_+^{q_1(x)} d\sigma \right) \right]^{\frac{\delta_1}{q_0^-(1-\hat{\eta})}} \right), \tag{3.25}$$

then (3.24) and in particular (3.23) are satisfied. Since $k_n \to 2k$ as $n \to \infty$ we obtain

$$\operatorname*{ess\,sup}_\Omega u \leq 2k = 2 \max\left(1, \left[(16K)^{\frac{1}{\delta_1}} b^{\frac{1}{\delta_1^2}} \left(\int_\Omega u_+^{q_0(x)} dx + \int_\Gamma u_+^{q_1(x)} d\sigma \right) \right]^{\frac{\delta_1}{q_0^-(1-\hat{\eta})}} \right).$$

Tracing back the constants, we see that the first part of the theorem is proved. The supersolution case can be done analogously, replacing u with $-u$ and A_k with $\tilde{A}_k$, and using Lemma 3.2 instead of Lemma 3.1. This completes the proof. □

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On the Boundedness of Solutions to Elliptic Variational Inequalities

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Abstract In this paper we present global a priori bounds for a class of variational inequalities involving general elliptic operators of second-order and terms of generalized directional derivatives. Based on Moser's and De Giorgi's iteration technique we prove the boundedness of solutions of such inequalities under certain criteria on the set of constraints. In our proofs we also use the localization method with a certain partition of unity and a version of a multiplicative inequality estimating the boundary integrals. Some sets of constraints satisfying the required conditions are stated as well.

Keywords A priori bounds · Clarke's gradient · De Giorgi iteration · Generalized directional derivative · Moser iteration · Variational-hemivariational inequality

Mathematics Subject Classifications (2010) 35B45 · 35J25 · 35J60 · 35J87

1 Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^N$, $N > 1$, with Lipschitz boundary $\partial\Omega$, we consider the following problem: Find $u \in K$ such that

$$\begin{aligned} &\int_\Omega A(x,u,\nabla u)\cdot\nabla(v-u)dx + \int_\Omega F(x,u,\nabla u)(v-u)dx \\ &+ \int_\Omega j_1^\circ(x,u;v-u)dx + \int_{\partial\Omega} j_2^\circ(x,u;v-u)d\sigma \geq 0, \end{aligned} \tag{1.1}$$

for all $v \in K$, where K is a subset of a Banach space V (will be specified below) and $j_k^\circ(x,s;r)$ $(k = 1,2)$ denotes the generalized directional derivative of a locally Lipschitz function $s \mapsto j_k(x,s)$ at s in the direction r. The maps $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are supposed to be Carathéodory functions satisfying suitable structure

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conditions (see hypotheses (H1) and (H2) below) while $d\sigma$ denotes the usual $(N-1)$-dimensional surface measure and for $u \in V$ the generalized boundary values on $\partial\Omega$ are well defined in the sense of traces. For reasons of simplification we drop the notation for the trace operator.

The aim of our treatment is to present conditions on the set of constraints, namely K, such that every solution $u \in K$ of (1.1) is essentially bounded. In order to specify the space V we will discuss two different cases: In Section 3 we choose $V = W^{1,p}(\Omega)$ with $1 < p < \infty$ and in Section 4 we set $V = W^{1,p(\cdot)}(\Omega)$ with $p \in C(\overline{\Omega})$ and $1 < \inf_{\overline{\Omega}} p$. These cases will be handled by different methods, the first one via Moser iteration and the second one via De Giorgi's iteration technique.

Due to the presence of the terms $j_k^\circ(x, s; r)$, inequalities of type (1.1) are called variational-hemivariational inequalities which include several interesting problems as special cases.

a) If $K = V$ and j_k $(k = 1, 2)$ are smooth, problem (1.1) becomes

$$\int_\Omega A(x, u, \nabla u) \cdot \nabla v dx + \int_\Omega F(x, u, \nabla u) v dx + \int_\Omega j_1'(x, u) v dx + \int_{\partial\Omega} j_2'(x, u) v d\sigma = 0, \quad \forall v \in V,$$

which means that $u \in V$ is a weak solution to the problem

$$\begin{aligned} -\operatorname{div} A(x, u, \nabla u) + F(x, u, \nabla u) + j_1'(x, u) &= 0 \qquad \text{in } \Omega, \\ A(x, u, \nabla u) \cdot \nu + j_2'(x, u) &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where $\nu(x)$ denotes the outer unit normal of Ω at $x \in \partial\Omega$. Regarding a priori bounds for such problems we refer to Winkert-Zacher [32, Theorem 1.1 and Corollary 1.2] (if $V = W^{1,p(\cdot)}(\Omega)$), Winkert [31, Theorem 4.1] (if $V = W^{1,p}(\Omega)$, see also Winkert [30, Proof of Proposition 5.2]) and Hu-Papageorgiou [18, Proposition 5] (if $V = W^{1,p}(\Omega)$ with homogeneous Neumann boundary condition).

b) If $j_k = 0$ $(k = 1, 2)$, (1.1) reduces to a classical variational inequality: Find $u \in K$ such that

$$\int_\Omega A(x, u, \nabla u) \cdot \nabla(v - u) dx + \int_\Omega F(x, u, \nabla u)(v - u) dx \geq 0 \quad \forall v \in K.$$

Boundedness results for solutions of such variational inequalities under suitable criteria on the set of constraints have been obtained by Kovalevsky-Nicolosi [20, Theorem 2.8], where $V = W^{1,p}(\theta, \Omega)$ with $1 < p < \infty$ and a positive weight θ satisfying $\theta \in L^1_{\text{loc}}(\Omega)$ and $1/\theta \in L^{1/(p-1)}_{\text{loc}}(\Omega)$ (see also Ježková [19] for local boundedness results). Concerning degenerated elliptic operators of high order we refer to a work of the same authors [21]. Recently, Gorban-Kovalevsky [16] have been studied the boundedness of solutions of degenerate anisotropic elliptic variational inequalities under certain conditions on the right-hand side and the set of constraints.

c) In case that K is the whole space V and j_k $(k = 1, 2)$ not necessarily smooth, problem (1.1) is a hemivariational inequality which contains as a special case the subsequent elliptic inclusion

$$\begin{aligned} -\operatorname{div} A(x, u, \nabla u) + F(x, u, \nabla u) + \partial j_1(x, u) &\ni 0 \qquad \text{in } \Omega, \\ A(x, u, \nabla u) \cdot \nu + \partial j_2(x, u) &\ni 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where the expression $\partial j_k(x, s)$ denotes the generalized gradient of the locally Lipschitz function $j_k(x, \cdot)$ in the sense of Clarke (see Section 2 for more details).

d) Let $V = W^{1,p}(\Omega)$ with $1 < p < \infty$. If the operator A satisfies appropriate monotonicity conditions with respect to the second and third argument and if the functions involved fulfill suitable structure conditions (similar to those in Section 3), then inequality (1.1) is equivalent to the multi-valued variational inequality: Find $u \in K$ such that

$$\begin{cases} \eta(x) \in \partial j_1(x, u(x)) \text{ a.e. in } \Omega, \xi(x) \in \partial j_2(x, u(x)) \text{ a.e. in } \partial\Omega, \\ \displaystyle\int_\Omega A(x, u, \nabla u) \cdot \nabla(v - u)dx + \int_\Omega F(x, u, \nabla u)(v - u)dx \\ \displaystyle + \int_\Omega \eta(v - u)dx + \int_{\partial\Omega} \xi(v - u)d\sigma \geq 0, \quad \forall v \in K. \end{cases}$$

This result was published by Carl [5].

We point out that our results are more general than those in [20]. On the one hand we extend their results to variational-hemivariational inequalities including nonlinear boundary terms and on the other hand we have weaker assumptions on the set of constraints. Indeed, hypothesis 2.6(i) in [20] is not needed in our treatment. For the sake of convenience we do not use Sobolev spaces with weights, but this case can be done in the same way. Another novelty of this work is the treatment of variational-hemivariational inequalities restricted to sets K belonging to variable exponent spaces $W^{1,p(\cdot)}(\Omega)$ with p continuous on $\overline{\Omega}$. To the best of our knowledge, our a priori estimates have not been published before.

Notice that we do not suppose that the set K is closed and convex in V. But in general, this is the typical assumption in the existence theory of inequalities like (1.1). Regarding existence and multiplicity results for problems of the form (1.1) we refer, without guarantee of completeness, to the papers of Carl [6], Kyritsi-Papageorgiou [23], Motreanu-Bonanno-Winkert [2], Bonanno-Winkert [3], Motreanu-Winkert [27] and the references therein. An overview about results to nonsmooth analysis and variational-hemivariational inequalities can be found in the monographs of Carl-Le-Motreanu [7] and Motreanu-Rădulescu [26]. We also point out a recent work of Carl [4] in which the class of variational-hemivariational inequalities has been extended to a more general class of inequalities.

The paper is organized as follows. In Section 2 we present some basic facts about nonsmooth analysis and the corresponding function spaces to problem (1.1). Furthermore, we prove an useful multiplicative inequality for boundary integrals. Section 3 handles the constant exponent case (i.e., $V = W^{1,p}(\Omega)$ with $1 < p < \infty$) where we will apply Moser's iteration following the ideas of Drábek-Kufner-Nicolosi [12]. In the last section we extend our results to the variable exponent case (i.e. $V = W^{1,p(\cdot)}(\Omega)$ with $p \in C(\overline{\Omega})$ and $1 < \inf_{\overline{\Omega}} p$) by applying De Giorgi's iteration. The results in this section are based on ideas of DiBenedetto [10], Ladyženskaja-Solonnikov-Ural'ceva [24], and Winkert-Zacher [32].

2 Preliminaries and Hypotheses

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ and let $p \in C(\overline{\Omega})$ with $p(x) > 1$ for all $x \in \overline{\Omega}$. Setting $p^- := \min_{x\in\overline{\Omega}} p(x)$ and $p^+ := \max_{x\in\overline{\Omega}} p(x)$, we have $p^- > 1$ and $p^+ < \infty$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u \;\middle|\; u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)}dx < +\infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\tau > 0 : \int_\Omega \left|\frac{u(x)}{\tau}\right|^{p(x)} dx \leq 1\right\}.$$

By $W^{1,p(\cdot)}(\Omega)$ we identify the variable exponent Sobolev space which is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{p(\cdot)}(\Omega)}.$$

We refer to the papers of Fan-Zhao [15], Kováčik-Rákosník [22] and the monograph of Diening-Harjulehto-Hästö-Růžička [11] for more information and basic properties about variable exponent spaces. If $p(x) \equiv p$ is a constant, the usual Sobolev space $W^{1,p}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx\right)^{\frac{1}{p}}.$$

Let us recall some basic facts on nonsmooth analysis. Let $(X, \|\cdot\|)$ be a real Banach space and denote by X^* its dual space while the duality pairing between X and X^* is denoted by $\langle\cdot,\cdot\rangle$. The dual space X^* is equipped with the dual norm $\|\cdot\|_*$, that is

$$\|\xi\|_* = \sup\{\langle\xi, v\rangle : v \in X, \|v\| \leq 1\}.$$

A function $j : X \to \mathbb{R}$ is said to be locally Lipschitz if for every $x \in X$ there exist a neighborhood U of x in X and a constant $C > 0$ such that

$$|j(y) - j(z)| \leq C\|y - z\|, \quad \text{for all } y, z \in U.$$

The generalized directional derivative of a locally Lipschitz function $j : X \to \mathbb{R}$ at a point $u \in X$ along the direction $v \in X$ is defined by

$$j^\circ(u; v) := \limsup_{x \to u, t \to 0^+} \frac{j(x + tv) - j(x)}{t}.$$

Since j is locally Lipschitz at u we have $j^\circ(u; v) \in \mathbb{R}$ for all $v \in X$. Furthermore, the function $j^\circ(u;\cdot) : X \to \mathbb{R}$ is subadditive, positively homogeneous and there holds the inequality

$$|j^\circ(u; v)| \leq C\|v\| \quad \text{for all } v \in X$$

with C being the Lipschitz constant of j near the point $u \in X$.

The generalized gradient of a locally Lipschitz function $j : X \to \mathbb{R}$ at a point $u \in X$, denoted by $\partial j(u)$, is the subset of X^* defined by

$$\partial j(u) := \left\{\xi \in X^* : j^\circ(u; v) \geq \langle\xi, v\rangle \text{ for all } v \in X\right\},$$

which is also known as Clarke's generalized gradient. Based on the Hahn-Banach theorem we easily verify that $\partial j(u)$ is nonempty. Moreover, $\partial j(u)$ is a convex, weak* compact subset of X^* and it holds $\|\xi\|_* \leq C$ for all $\xi \in \partial j(u)$. For every $v \in X$, one has

$$j^\circ(u; v) = \max\{\langle\xi, v\rangle : \xi \in \partial j(u)\}.$$

We refer to the monographs of Clarke [9] and Motreanu-Rădulescu [26] as well as the paper of Chang [8] for more details and properties of generalized directional derivatives and generalized gradients.

The next result is needed for the proof via Moser iteration to estimate the boundary integrals by suitable integrals defined in Ω. The proof is based on appropriate embeddings and interpolation results of Besov and Sobolev Slobodeckij spaces.

Proposition 2.1 *Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$, let $1 < p < \infty$, and let q be such that $p \le q < p_*$ with the critical exponent*

$$p_* = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N, \\ \infty & \text{if } p \ge N. \end{cases}$$

Then, for every $\varepsilon > 0$, there exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$\|u\|^p_{L^q(\partial\Omega)} \le \varepsilon \|u\|^p_{W^{1,p}(\Omega)} + a_1 \varepsilon^{-a_2} \|u\|^p_{L^p(\Omega)} \qquad \text{for all } u \in W^{1,p}(\Omega).$$

Proof Since $q < p_*$ we may fix a number $\theta \in (0, 1)$ small enough such that

$$q \begin{cases} \le \frac{(N-1)p}{N-p+\theta p} & \text{if } p \le N \\ < \frac{p-N}{\theta} & \text{if } p > N \end{cases} \quad \text{and} \quad 1 - \theta > \frac{1}{p}. \tag{2.1}$$

From Triebel [28, 3.3.3], we have the continuous embedding

$$B^{1-\theta}_{p,p}(\Omega) = W^{1-\theta,p}(\Omega) \to B^{1-\theta-\frac{1}{p}}_{p,p}(\partial\Omega) = W^{1-\theta-\frac{1}{p},p}(\partial\Omega), \tag{2.2}$$

where $B^s_{p,p}$, $s \in (0, 1)$, denotes the Besov space which coincides with the Sobolev Slobodeckij space $W^{s,p}$. Note that the embedding (2.2) requires only a Lipschitz boundary as $1 - \theta < 1$.

From the choice of $\theta \in (0, 1)$ and since $p \le q$ (see also (2.1)) we get

$$\left(1 - \theta - \frac{1}{p}\right) p \begin{cases} < N - 1 & \text{if } p \le N, \\ > N - 1 & \text{if } p > N. \end{cases}$$

Taking into account the Sobolev embedding theorem for fractional order Sobolev spaces (see Adams [1, Theorem 7.57]) gives

$$W^{1-\theta-\frac{1}{p},p}(\partial\Omega) \to L^q(\partial\Omega) \tag{2.3}$$

for

$$q \begin{cases} \le \frac{(N-1)p}{N-1-\left(1-\theta-\frac{1}{p}\right)p} = \frac{(N-1)p}{N-p+\theta p} & \text{if } \left(1 - \theta - \frac{1}{p}\right) p < N - 1, \\ < \infty & \text{if } \left(1 - \theta - \frac{1}{p}\right) p > N - 1. \end{cases}$$

Actually, in case $\left(1 - \theta - \frac{1}{p}\right) p > N - 1$ we have the stronger embedding $W^{1-\theta-\frac{1}{p},p}(\partial\Omega) \to C(\partial\Omega)$.

Since $W^{1,p}(\Omega) \subset W^{1-\theta,p}(\Omega) \subset L^p(\Omega)$ are continuous embeddings we may apply real interpolation (see Triebel [29, 1.6.2 and 1.6.7])

$$(L^p(\Omega), W^{1,p}(\Omega))_{1-\theta,p} = W^{1-\theta,p}(\Omega),$$

which implies the estimate

$$\|u\|_{W^{1-\theta,p}(\Omega)} \le \tilde{C}_1 \|u\|^{1-\theta}_{W^{1,p}(\Omega)} \|u\|^{1-(1-\theta)}_{L^p(\Omega)} \qquad \text{for all } u \in W^{1,p}(\Omega) \tag{2.4}$$

with a positive constant $\tilde{C}_1$. Combining (2.2)–(2.4) and using Young's inequality with $\tilde{\delta} > 0$ results in

$$\begin{aligned}\|u\|^p_{L^q(\partial\Omega)} &\leq \tilde{C}_2 \tilde{\delta}^{1-\theta} \|u\|^{(1-\theta)p}_{W^{1,p}(\Omega)} \tilde{\delta}^{-1+\theta} \|u\|^{\theta p}_{L^p(\Omega)} \\ &\leq \tilde{C}_2 \left(\tilde{\delta} \|u\|^p_{W^{1,p}(\Omega)} + \tilde{\delta}^{\frac{-1+\theta}{\theta}} \|u\|^p_{L^p(\Omega)} \right).\end{aligned}$$

Setting $\tilde{\delta} := \frac{\varepsilon}{\tilde{C}_2}$ with arbitrary $\varepsilon > 0$ provides the desired estimate. □

Finally, we conclude this section by fixing our notation. If $s \in \mathbb{R}$ we write $s_+ = \max(s, 0)$ and $s_- = \min(s, 0)$. For functions $u, v : \Omega \to \mathbb{R}$ we use the notation $u \wedge v = \min(u, v)$, $u \vee v = \max(u, v)$, $K \wedge K = \{u \wedge v : u, v \in K\}$, and $K \vee K = \{u \vee v : u, v \in K\}$. If the set $K \subseteq V$ satisfies

$$K \wedge K \subset K \quad \text{and} \quad K \vee K \subset K, \tag{2.5}$$

we say that K has lattice structure. Note that V has lattice structure, that means,

$$V \wedge V \subset V \quad \text{and} \quad V \vee V \subset V$$

(see Heinonen-Kilpeläinen-Martio [17, Theorem 1.20] if $V = W^{1,p}(\Omega)$ and Le [25, Lemma 3.2] if $V = W^{1,p(\cdot)}(\Omega)$). Throughout the paper we will denote by M_i and $\hat{M}_j$, $i, j = 1, 2, \ldots$ positive constants depending on the given data and the Lebesgue measure on $\mathbb{R}^N$ is given by $|\cdot|_N$.

3 The Case $V = W^{1,p}(\Omega)$ Via Moser Iteration

We start our treatment with the constant exponent case and use Moser's iteration to prove L^∞-bounds for solutions of inequality (1.1). In this section we suppose the following assumptions.

(H1) The mappings $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are supposed to satisfy a Carathéodory condition while $x \mapsto j_1(x, s)$, $x \mapsto j_2(x, s)$ are measurable for all $s \in \mathbb{R}$ and $s \mapsto j_1(x, s)$, $s \mapsto j_2(x, s)$ are locally Lipschitz for a.a. $x \in \Omega$ and for a.a. $x \in \partial\Omega$, respectively. In addition the subsequent structure conditions are assumed:

(i) $A(x, s, \xi) \cdot \xi \geq a_1 |\xi|^p - a_2 |s|^{q_1} - a_3$ for a.a. $x \in \Omega$;

(ii) $|A(x, s, \xi)| \leq a_4 |\xi|^{p-1} + a_5 |s|^{q_1 \frac{p-1}{p}} + a_6$ for a.a. $x \in \Omega$;

(iii) $|F(x, s, \xi)| \leq b_1 |\xi|^{p \frac{q_1 - 1}{q_1}} + b_2 |s|^{q_1 - 1} + b_3$ for a.a. $x \in \Omega$;

(iv) $|\eta| \leq c_1 |s|^{q_1 - 1} + c_2$ for a.a. $x \in \Omega$, for all $\eta \in \partial j_1(x, s)$;

(v) $|\tau| \leq d_1 |s|^{q_2 - 1} + d_2$, for a.a. $x \in \partial\Omega$, for all $\tau \in \partial j_2(x, s)$;

and for all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^N$ with positive constants a_i, b_j, c_k, d_l ($i \in \{1, \ldots, 6\}$, $j \in \{1, \ldots, 3\}$, $k, l \in \{1, 2\}$) and fixed numbers p, q_1, q_2 such that

$$1 < p < \infty, \qquad p \leq q_1 < p^*, \qquad p \leq q_2 < p_*$$

with the critical exponents

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N, \end{cases} \qquad p_* = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

Recalling

$$\partial j_k(x,s) = \left\{\xi \in \mathbb{R} : j_k^\circ(x,s;r) \geq \xi r, \forall r \in \mathbb{R}\right\}$$

for a.a. $x \in \Omega$ ($k=1$), respectively, for a.a. $x \in \partial\Omega$ ($k=2$) we easily derive the estimates

$$|j_1^\circ(x,s;r)| \leq \left(c_1|s|^{q_1-1} + c_2\right)|r| \qquad \text{for a.a. } x \in \Omega \text{ and for all } s, r \in \mathbb{R}, \tag{3.1}$$

$$|j_2^\circ(x,s;r)| \leq \left(d_1|s|^{q_2-1} + d_2\right)|r| \qquad \text{for a.a. } x \in \partial\Omega \text{ and for all } s, r \in \mathbb{R}, \tag{3.2}$$

thanks to (H1)(iv) and (v). From the embeddings $W^{1,p}(\Omega) \to L^{q_1}(\Omega)$ and $W^{1,p}(\Omega) \to L^{q_2}(\partial\Omega)$ (see Adams [1]) and the growth conditions in (H1)(ii), (iii), (3.1) as well as (3.2) we see that the integrals in (1.1) are finite.

For $u \in W^{1,p}(\Omega)$ and $\alpha, \beta > 0$ we define the functions $v_\alpha(x) := \min(u_+(x), \alpha)$ and $v_\beta(x) := \max(u_-(x), -\beta)$ which belong both to $W^{1,p}(\Omega)$. We suppose the following conditions on the set K.

(K1) For $u \in K$, $\alpha > 0$, and $\kappa > 0$ there exists $t > 0$ such that

$$\varphi = u - t v_\alpha^{\kappa p} u \in K.$$

(K2) For $u \in K$, $\beta > 0$, and $\kappa > 0$ there exists $h > 0$ such that

$$\psi = u - h(-v_\beta)^{\kappa p} u \in K.$$

Note that both φ and ψ are elements of $W^{1,p}(\Omega)$ and their gradients are given by

$$\nabla\varphi = \nabla u - t\kappa p v_\alpha^{\kappa p-1}\nabla v_\alpha u - t v_\alpha^{\kappa p}\nabla u = \nabla u - t\kappa p v_\alpha^{\kappa p}\nabla v_\alpha - t v_\alpha^{\kappa p}\nabla u$$

respectively,

$$\begin{aligned}\nabla\psi &= \nabla u - h\kappa p(-v_\beta)^{\kappa p-1}\nabla(-v_\beta)u - h(-v_\beta)^{\kappa p}\nabla u \\ &= \nabla u + h\kappa p(-v_\beta)^{\kappa p}\nabla(-v_\beta) - h(-v_\beta)^{\kappa p}\nabla u.\end{aligned}$$

We have the following result.

Theorem 3.1 *Let hypothesis (H1) be satisfied and let $u \in K$ be a solution of (1.1). Then there exists a constant $C_1 > 0$ such that the following assertions hold.*

(1) *If condition (K1) is satisfied, then*

$$\operatorname*{ess\,sup}_{x\in\Omega} u(x) \leq C_1.$$

(2) *If condition (K2) is satisfied, then*

$$\operatorname*{ess\,inf}_{x\in\Omega} u(x) \geq -C_1.$$

Proof We start with (1) and assume, without loss of generality, that $p < q_1$ and $p < q_2$. The cases $p = q_1$ and/or $p = q_2$ work similarly. Thanks to (K1) we may take $v = \varphi = u - t v_\alpha^{\kappa p} u \in K$ in (1.1). This gives

$$\begin{aligned}&t\kappa p \int_\Omega A(x,u,\nabla u)\cdot\nabla v_\alpha v_\alpha^{\kappa p} dx + t\int_\Omega A(x,u,\nabla u)\cdot\nabla u v_\alpha^{\kappa p} dx \\ &\leq -t\int_\Omega F(x,u,\nabla u) v_\alpha^{\kappa p} u dx + \int_\Omega j_1^\circ\left(x,u;-t v_\alpha^{\kappa p} u\right) dx \\ &\quad + \int_{\partial\Omega} j_2^\circ\left(x,u;-t v_\alpha^{\kappa p} u\right) d\sigma.\end{aligned} \tag{3.3}$$

Since j_k $(k = 1, 2)$ are positively homogeneous with respect to the third argument (cf. [9, Proposition 2.1.1]), we can divide inequality (3.3) by $t > 0$.

Let us estimate the several terms in (3.3). We start with the left-hand side of (3.3) by applying hypothesis (H1)(i) and Hölder's inequality. We obtain

$$\begin{aligned}
&\kappa p \int_\Omega A(x,u,\nabla u)\cdot\nabla v_\alpha v_\alpha^{\kappa p} dx + \int_\Omega A(x,u,\nabla u)\cdot\nabla u v_\alpha^{\kappa p} dx \\
&= \kappa p \int_\Omega A(x,v_\alpha,\nabla v_\alpha)\cdot\nabla v_\alpha v_\alpha^{\kappa p} dx + \int_\Omega A(x,u_+,\nabla u_+)\cdot\nabla u_+ v_\alpha^{\kappa p} dx \\
&\geq \kappa p \int_\Omega \left(a_1|\nabla v_\alpha|^p - a_2|v_\alpha|^{q_1} - a_3\right) v_\alpha^{\kappa p} dx \\
&\quad + \int_\Omega \left(a_1|\nabla u_+|^p - a_2|u_+|^{q_1} - a_3\right) v_\alpha^{\kappa p} dx \\
&\geq \kappa p \left(a_1 \int_\Omega |\nabla v_\alpha|^p v_\alpha^{\kappa p} dx - a_2 \int_\Omega u_+^{(\kappa+1)p} u_+^{q_1-p} dx - a_3 \int_\Omega u_+^{\kappa p} dx\right) \\
&\quad + a_1 \int_\Omega |\nabla u_+|^p v_\alpha^{\kappa p} dx - (a_2+a_3) \int_\Omega u_+^{(\kappa+1)p} u_+^{q_1-p} dx - a_3|\Omega|_N \\
&\geq a_1 \kappa p \int_\Omega |\nabla v_\alpha|^p \, v_\alpha^{\kappa p} dx + a_1 \int_\Omega |\nabla u_+|^p v_\alpha^{\kappa p} dx \\
&\quad - M_1(\kappa p + 1) \left\| u_+^{\kappa+1} \right\|_{L^{q_1}(\Omega)}^p - M_2.
\end{aligned} \tag{3.4}$$

By means of (H1)(iii) combined with Hölder's inequality and Young's inequality with $\varepsilon_1 > 0$ the first term on the right-hand side can be estimated through

$$\begin{aligned}
&-\int_\Omega F(x,u,\nabla u) v_\alpha^{\kappa p} u dx \\
&\leq b_1 \int_\Omega \varepsilon_1^{\frac{q_1-1}{q_1}} |\nabla u_+|^{p\frac{q_1-1}{q_1}} u_+^{\kappa p \frac{q_1-1}{q_1}} \varepsilon_1^{-\frac{q_1-1}{q_1}} u_+^{\kappa p\left(1-\frac{q_1-1}{q_1}\right)+1} dx \\
&\quad + (b_2+b_3) \int_\Omega u_+^{(\kappa+1)p} u_+^{q_1-p} dx + b_3|\Omega|_N \\
&\leq \frac{\varepsilon_1 b_1}{(\kappa+1)^p} \int_\Omega |\nabla u_+^{\kappa+1}|^p dx + M_3\left(\varepsilon_1^{-(q_1-1)} + 1\right) \left\| u_+^{\kappa+1} \right\|_{L^{q_1}(\Omega)}^p + M_4.
\end{aligned} \tag{3.5}$$

Owing to (3.1) combined with Hölder's inequality the second integral on the right-hand side of (3.3) gives

$$\begin{aligned}
\int_\Omega j_1^\circ\left(x,u;-v_\alpha^{\kappa p}u\right) dx &\leq \int_\Omega \left(c_1|u|^{q_1-1} + c_2\right) u_+^{\kappa p+1} dx \\
&\leq M_5 \left\| u_+^{\kappa+1} \right\|_{L^{q_1}(\Omega)}^p + M_6.
\end{aligned} \tag{3.6}$$

In the same way, using (3.2) and again Hölder's inequality, we get for the last integral

$$\begin{aligned}
\int_{\partial\Omega} j_2^\circ\left(x,u;-v_\alpha^{\kappa p}u\right) d\sigma &\leq \int_{\partial\Omega} \left(d_1|u|^{q_2-1} + d_2\right) u_+^{\kappa p+1} d\sigma \\
&\leq M_7 \left\| u_+^{\kappa+1} \right\|_{L^{q_2}(\partial\Omega)}^p + M_8.
\end{aligned} \tag{3.7}$$

Now, combining (3.3)–(3.7) we have

$$\begin{aligned} & a_1 \kappa p \int_\Omega |\nabla v_\alpha|^p \, v_\alpha^{\kappa p} dx + a_1 \int_\Omega |\nabla u_+|^p v_\alpha^{\kappa p} dx \\ & \le \frac{\varepsilon_1 b_1}{(\kappa+1)^p} \int_\Omega |\nabla u_+^{\kappa+1}|^p dx + M_9 \left(\varepsilon_1^{-(q_1-1)} + 1 \right) \left\| u_+^{\kappa+1} \right\|_{L^{q_1}(\Omega)}^p \\ & \quad + M_{10} \left\| u_+^{\kappa+1} \right\|_{L^{q_2}(\partial\Omega)}^p + M_{11}. \end{aligned}$$

Taking into account Fatou's lemma yields

$$\begin{aligned} & a_1 \frac{\kappa p+1}{(\kappa+1)^p} \int_\Omega \left| \nabla u_+^{\kappa+1} \right|^p dx \\ & \le \frac{\varepsilon_1 b_1}{(\kappa+1)^p} \int_\Omega |\nabla u_+^{\kappa+1}|^p dx + M_9 \left(\varepsilon_1^{-(q_1-1)} + 1 \right) \left\| u_+^{\kappa+1} \right\|_{L^{q_1}(\Omega)}^p \\ & \quad + M_{10} \left\| u_+^{\kappa+1} \right\|_{L^{q_2}(\partial\Omega)}^p + M_{11}. \end{aligned} \tag{3.8}$$

Choosing $\varepsilon_1 = \frac{a_1}{2b_1}(\kappa p + 1)$ implies $\varepsilon_1^{-(q_1-1)} \le M_{12}$. Then, (3.8) becomes

$$\begin{aligned} & \frac{a_1}{2} \frac{\kappa p+1}{(\kappa+1)^p} \int_\Omega \left| \nabla u_+^{\kappa+1} \right|^p dx \\ & \le M_{13} \left\| u_+^{\kappa+1} \right\|_{L^{q_1}(\Omega)}^p + M_{10} \left\| u_+^{\kappa+1} \right\|_{L^{q_2}(\partial\Omega)}^p + M_{11}. \end{aligned} \tag{3.9}$$

Now, we may apply Proposition 2.1 to the boundary term in (3.9) (with $q = q_2$, $\varepsilon = \varepsilon_2$) and Hölder's inequality as before to get

$$\begin{aligned} & \left\| u_+^{\kappa+1} \right\|_{L^{q_2}(\partial\Omega)}^p \\ & \le \varepsilon_2 \int_\Omega |\nabla u_+^{\kappa+1}|^p dx + \varepsilon_2 \int_\Omega u_+^{(\kappa+1)p} dx + a_1 \varepsilon_2^{-a_2} \int_\Omega u_+^{(\kappa+1)p} dx \\ & \le \varepsilon_2 \int_\Omega |\nabla u_+^{\kappa+1}|^p dx + M_{14} \left(\varepsilon_2 + \varepsilon_2^{-a_2} \right) \left\| u_+^{\kappa+1} \right\|_{L^{q_1}(\Omega)}^p. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10) results in

$$\begin{aligned} & \left(\frac{a_1}{2} \frac{\kappa p+1}{(\kappa+1)^p} - M_{10}\varepsilon_2 \right) \int_\Omega \left| \nabla u_+^{\kappa+1} \right|^p dx \\ & \le M_{15} \left(\varepsilon_2 + \varepsilon_2^{-a_2} + 1 \right) \left\| u_+^{\kappa+1} \right\|_{L^{q_1}(\Omega)}^p + M_{11}. \end{aligned}$$

The choice $\varepsilon_2 = \frac{a_1}{4M_{10}} \frac{\kappa p+1}{(\kappa+1)^p}$ gives

$$\frac{a_1}{4} \frac{\kappa p+1}{(\kappa+1)^p} \int_\Omega \left| \nabla u_+^{\kappa+1} \right|^p dx \le M_{16} \left(\frac{(\kappa+1)^p}{\kappa p+1} \right)^{M_{17}} \left\| u_+^{\kappa+1} \right\|_{L^{q_1}(\Omega)}^p + M_{11}. \tag{3.11}$$

Finally, we divide (3.11) by $\frac{a_1}{4} \frac{\kappa p+1}{(\kappa+1)^p} > 0$ and add on both sides the integral $\int_\Omega u_+^{(\kappa+1)p} dx$ which leads to

$$\begin{aligned} & \int_\Omega \left| \nabla u_+^{\kappa+1} \right|^p dx + \int_\Omega u_+^{(\kappa+1)p} dx \\ & \le M_{18} \left(\frac{(\kappa+1)^p}{\kappa p+1} \right)^{M_{19}} \left(\left\| u_+^{\kappa+1} \right\|_{L^{q_1}(\Omega)}^p + 1 \right), \end{aligned} \tag{3.12}$$

where Hölder's inequality was again taken into account on the right-hand side.

Having regard to the continuous embedding $W^{1,p}(\Omega) \to L^{\tilde{p}^*}(\Omega)$ with

$$\tilde{p}^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ 2q_1 & \text{if } p \geq N, \end{cases}$$

we obtain

$$\begin{aligned} \|u_+\|_{L^{(\kappa+1)\tilde{p}^*}(\Omega)} &= \left\|u_+^{\kappa+1}\right\|_{L^{\tilde{p}^*}(\Omega)}^{\frac{1}{\kappa+1}} \\ &\leq M_{20}^{\frac{1}{\kappa+1}} \|u_+^{\kappa+1}\|_{W^{1,p}(\Omega)}^{\frac{1}{\kappa+1}} \\ &= M_{20}^{\frac{1}{\kappa+1}} \left(\int_\Omega |\nabla u_+^{\kappa+1}|^p dx + \int_\Omega u_+^{(\kappa+1)p} dx\right)^{\frac{1}{(\kappa+1)p}}, \end{aligned} \tag{3.13}$$

where $M_{20} > 0$ denotes the embedding constant. Combining (3.12) and (3.13) we derive

$$\begin{aligned} &\|u_+\|_{L^{(\kappa+1)\tilde{p}^*}(\Omega)} \\ &\leq M_{20}^{\frac{1}{\kappa+1}} \left(\int_\Omega |\nabla u_+^{\kappa+1}|^p dx + \int_\Omega u_+^{(\kappa+1)p} dx\right)^{\frac{1}{(\kappa+1)p}} \\ &\leq M_{20}^{\frac{1}{\kappa+1}} M_{18}^{\frac{1}{(\kappa+1)p}} \left(\frac{(\kappa+1)^{M_{19}}}{(\kappa p+1)^{\frac{M_{19}}{p}}}\right)^{\frac{1}{\kappa+1}} \left(\left\|u_+^{\kappa+1}\right\|_{L^{q_1}(\Omega)}^p + 1\right)^{\frac{1}{(\kappa+1)p}}. \end{aligned} \tag{3.14}$$

Observe that

$$\left(\frac{(\kappa+1)^{M_{19}}}{(\kappa p+1)^{\frac{M_{19}}{p}}}\right)^{\frac{1}{\sqrt{\kappa+1}}} \geq 1 \qquad \text{and} \qquad \lim_{\kappa\to\infty} \left(\frac{(\kappa+1)^{M_{19}}}{(\kappa p+1)^{\frac{M_{19}}{p}}}\right)^{\frac{1}{\sqrt{\kappa+1}}} = 1.$$

Hence, we find a constant $M_{21} > 1$ such that

$$\left(\frac{(\kappa+1)^{M_{19}}}{(\kappa p+1)^{\frac{M_{19}}{p}}}\right)^{\frac{1}{\kappa+1}} \leq M_{21}^{\frac{1}{\sqrt{\kappa+1}}}. \tag{3.15}$$

Applying (3.15) to (3.14) gives the estimate

$$\|u_+\|_{L^{(\kappa+1)\tilde{p}^*}(\Omega)} \leq M_{20}^{\frac{1}{\kappa+1}} M_{18}^{\frac{1}{(\kappa+1)p}} M_{21}^{\frac{1}{\sqrt{\kappa+1}}} \left(\left\|u_+^{\kappa+1}\right\|_{L^{q_1}(\Omega)}^p + 1\right)^{\frac{1}{(\kappa+1)p}}. \tag{3.16}$$

Now we can start with the typical bootstrap arguments. Choosing κ such that

$$\begin{aligned} &\kappa_1 : (\kappa_1+1)q_1 = \tilde{p}^*, \\ &\kappa_2 : (\kappa_2+1)q_1 = (\kappa_1+1)\tilde{p}^*, \\ &\kappa_3 : (\kappa_3+1)q_1 = (\kappa_2+1)\tilde{p}^*, \\ &\quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad . \end{aligned}$$

we see that

$$\|u_+\|_{L^{(\kappa+1)\tilde{p}^*}(\Omega)} \leq C(\kappa)$$

for any finite number κ, where $C(\kappa)$ is a positive constant depending on κ. Thus, $u_+ \in L^r(\Omega)$ for any $r \in (1,\infty)$.

In order to prove the uniform estimate with respect to κ we argue as follows. If there is a sequence $\kappa_n \to \infty$ such that

$$\int_\Omega u_+^{(\kappa_n+1)q_1}\,dx \le 1,$$

we directly obtain

$$\|u_+\|_{L^\infty(\Omega)} \le 1.$$

In the opposite case there exists a number $\kappa_0 > 0$ such that

$$\int_\Omega u_+^{(\kappa+1)q_1}\,dx > 1 \qquad \text{for any } \kappa \ge \kappa_0.$$

Then we conclude from (3.16)

$$\begin{aligned}\|u_+\|_{L^{(\kappa+1)\tilde p^*}(\Omega)} &\le M_{20}^{\frac{1}{\kappa+1}} M_{18}^{\frac{1}{(\kappa+1)p}} M_{21}^{\frac{1}{\sqrt{\kappa+1}}} \left(2\left\|u_+^{\kappa+1}\right\|^p_{L^{q_1}(\Omega)}\right)^{\frac{1}{(\kappa+1)p}} \\ &\le M_{20}^{\frac{1}{\kappa+1}} M_{22}^{\frac{1}{(\kappa+1)p}} M_{21}^{\frac{1}{\sqrt{\kappa+1}}} \|u_+\|_{L^{(\kappa+1)q_1}(\Omega)}.\end{aligned} \tag{3.17}$$

Applying again the bootstrap arguments we define a sequence (κ_n) such that

$$\begin{aligned}&\kappa_1 : (\kappa_1+1)q_1 = (\kappa_0+1)\tilde p^*,\\ &\kappa_2 : (\kappa_2+1)q_1 = (\kappa_1+1)\tilde p^*,\\ &\kappa_3 : (\kappa_3+1)q_1 = (\kappa_2+1)\tilde p^*,\\ &\quad\vdots \qquad\qquad\quad \vdots \qquad\qquad .\end{aligned} \tag{3.18}$$

By induction, from (3.17) and (3.18), we obtain

$$\begin{aligned}\|u_+\|_{L^{(\kappa_n+1)\tilde p^*}(\Omega)} &\le M_{20}^{\frac{1}{\kappa_n+1}} M_{22}^{\frac{1}{(\kappa_n+1)p}} M_{21}^{\frac{1}{\sqrt{\kappa_n+1}}} \|u_+\|_{L^{(\kappa_n+1)q_1}(\Omega)} \\ &= M_{20}^{\frac{1}{\kappa_n+1}} M_{22}^{\frac{1}{(\kappa_n+1)p}} M_{21}^{\frac{1}{\sqrt{\kappa_n+1}}} \|u_+\|_{L^{(\kappa_{n-1}+1)\tilde p^*}(\Omega)}\end{aligned}$$

for any $n \in \mathbb{N}$, where the sequence (κ_n) is chosen such that $(\kappa_n+1) = (\kappa_0+1)\left(\frac{\tilde p^*}{q_1}\right)^n$. Then, we have

$$\|u_+\|_{L^{(\kappa_n+1)\tilde p^*}(\Omega)} \le M_{20}^{\sum_{i=1}^n \frac{1}{\kappa_i+1}} M_{22}^{\sum_{i=1}^n \frac{1}{(\kappa_i+1)p}} M_{21}^{\sum_{i=1}^n \frac{1}{\sqrt{\kappa_i+1}}} \|u_+\|_{L^{(\kappa_0+1)\tilde p^*}(\Omega)},$$

with $(\kappa_n+1)\tilde p^* \to \infty$ as $n \to \infty$. Since $\frac{1}{\kappa_i+1} = \frac{1}{\kappa_0+1}\left(\frac{q_1}{\tilde p^*}\right)^i$ and $\frac{q_1}{\tilde p^*} < 1$ there is a constant $M_{23} > 0$ such that

$$\|u_+\|_{L^{(\kappa_n+1)\tilde p^*}(\Omega)} \le M_{23}\|u_+\|_{L^{(k_0+1)\tilde p^*}(\Omega)}. \tag{3.19}$$

Since $u_+ \in L^r(\Omega)$ for any $r \in (1,\infty)$, the right-hand side of (3.19) is finite. By means of (3.19) it follows that

$$\operatorname*{ess\,sup}_{x\in\Omega} u_+(x) \le C_1$$

(see Drábek-Kufner-Nicolosi [12, proof of Lemma 3.2]).

The second part can be proved similarly. We take $v = \psi = u - h(-v_\beta)^{\kappa p}u$ in (1.1) which leads to

$$\begin{aligned} &h\kappa p \int_\Omega A(x,u,\nabla u)\cdot\nabla v_\beta(-v_\beta)^{\kappa p}dx + h\int_\Omega A(x,u,\nabla u)\cdot\nabla u(-v_\beta)^{\kappa p}dx \\ &\leq -h\int_\Omega F(x,u,\nabla u)(-v_\beta)^{\kappa p}u dx + \int_\Omega j_1^\circ\left(x,u;-h(-v_\beta)^{\kappa p}u\right)dx \\ &\quad + \int_{\partial\Omega} j_2^\circ\left(x,u;-h(-v_\beta)^{\kappa p}u\right)d\sigma. \end{aligned} \tag{3.20}$$

Applying again the structure conditions in (H1) combined with Hölder's and Young's inequality to (3.20) the statement in (2) can be shown as in the first part. This completes the proof. □

As a direct consequence of Theorem 3.1 we obtain the following corollary.

Corollary 3.1 *Assume hypotheses (H1), (K1), (K2) and let $u \in K$ be a solution of (1.1). Then there holds*

$$\operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \leq C_1,$$

where the constant C_1 is the same as in Theorem 3.1.

Remark 3.3 Note that the constant C_1 in Theorem 3.1 depends on the given data and on the solution $u \in K$, that is,

$$C_1 = C_1(p, q_1, q_2, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, d_1, d_2, \Omega, u).$$

The assumption (H1)(ii) is not needed in the proof of Theorem 3.1. It should be noted that the finiteness of the integrals

$$\int_\Omega |\nabla u_+^{\kappa+1}|^p dx, \quad \int_\Omega u_+^{(\kappa+1)q_1}dx$$

is shown in the end of the proof of Theorem 3.1 by a suitable choice of the parameter κ. This is a typical proceeding in the usage of the Moser iteration.

Remark 3.4 If $V = W_0^{1,p}(\Omega)$ being the usual Sobolev space consisting of the elements of $W^{1,p}(\Omega)$ with zero traces on $\partial\Omega$, problem (1.1) reduces to

$$\begin{aligned} u \in K: \quad &\int_\Omega A(x,u,\nabla u)\cdot\nabla(v-u)dx + \int_\Omega F(x,u,\nabla u)(v-u)dx \\ &+ \int_\Omega j_1^\circ(x,u;v-u)dx \geq 0, \end{aligned} \tag{3.21}$$

for all $v \in K \subseteq W_0^{1,p}(\Omega)$. We observe that the results ensured by Theorem 3.1 still hold for problems of type (3.21). In this case we do not need Proposition 2.1 and the proof becomes more simple. In general, the assumptions in (K1) and (K2) are satisfied if $K = W_0^{1,p}(\Omega)$ and $K = W^{1,p}(\Omega)$.

Remark 3.5 As already mentioned in the Introduction our assumptions on the set of constraints are weaker than those in Kovalevsky-Nikolosi [20]. Precisely, Hypothesis 2.6(i) in [20] is not necessary in our proof.

Let us consider some examples of suitable sets K. To this end, let $\vartheta, \vartheta_j, \omega, \psi : \Omega \to \mathbb{R}$, $j = 1, \ldots, N$, be given functions such that $\vartheta \geq 0$, $\omega \leq 0 \leq \psi$ a.e. in Ω.

Example 3.6 Let $\Theta : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $s \to \Theta(x, s)$ is convex for a.a. $x \in \Omega$ and such that $\Theta(x, 0) \leq 0$ for a.a. $x \in \Omega$. Then the set

$$K = \left\{ u \in W^{1,p}(\Omega) : \Theta(x, u(x)) \leq 0 \quad \text{a.e. in } \Omega \right\}$$

fulfills the conditions (K1), (K2) with $t = \alpha^{-\kappa p}$ and $h = \beta^{-\kappa p}$. In particular, Θ can be of the following form

$$\Theta(x, s) = s - \vartheta(x), \qquad \Theta(x, s) = -s - \vartheta(x), \qquad \Theta(x, s) = (s - \omega(x))(s - \psi(x)).$$

From this choice we see that the subsequent sets satisfy our assumptions:

$$K = \{u \in W^{1,p}(\Omega) : u \leq \vartheta \quad \text{a.e. in } \Omega\},$$
$$K = \{u \in W^{1,p}(\Omega) : u \geq -\vartheta \quad \text{a.e. in } \Omega\},$$
$$K = \{u \in W^{1,p}(\Omega) : \omega \leq u \leq \psi \quad \text{a.e. in } \Omega\}.$$

Example 3.7 Let $\Upsilon : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a function such that $\xi \to \Upsilon(x, \xi)$ is convex for a.a. $x \in \Omega$ and such that $\Upsilon(x, 0) \leq 0$ for a.a. $x \in \Omega$. Then the set

$$K = \left\{ u \in W^{1,p}(\Omega) : \Upsilon(x, \nabla u(x)) \leq 0 \quad \text{a.e. in } \Omega \right\}$$

fulfills the conditions (K1), (K2) with $t = (\kappa p + 1)^{-1} \alpha^{-\kappa p}$ and $h = (\kappa p + 1)^{-1} \beta^{-\kappa p}$. As before, we see that

$$\Upsilon(x, \xi) = |\xi| - \vartheta(x), \qquad \Upsilon(x, \xi) = \sum_{j=1}^{N} \vartheta_j(x) \xi_j - \vartheta(x), \quad \xi = (\xi_1, \ldots, \xi_N)$$

are suitable choices for Υ. Hence, our results can be applied to the sets

$$K = \{u \in W^{1,p}(\Omega) : |\nabla u| \leq \vartheta \quad \text{a.e. in } \Omega\},$$
$$K = \left\{ u \in W^{1,p}(\Omega) : \sum_{j=1}^{N} \vartheta_j \frac{\partial u}{\partial x_j} \leq \vartheta \quad \text{a.e. in } \Omega \right\}.$$

4 The Case $V = W^{1,p(\cdot)}(\Omega)$ Via De Giorgi Iteration

In this section we will provide a priori bounds for (1.1) by applying De Giorgi's iteration technique if p is a continuous function on $\overline{\Omega}$. Our assumptions in that case read as follows:

(H2) The mappings $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are supposed to satisfy a Carathéodory condition while $x \mapsto j_1(x, s)$, $x \mapsto j_2(x, s)$ are measurable for all $s \in \mathbb{R}$ and $s \mapsto j_1(x, s)$, $s \mapsto j_2(x, s)$ are locally Lipschitz for a.a. $x \in \Omega$ and for a.a. $x \in \partial\Omega$, respectively. Furthermore, it is assumed the following:

(i) $A(x, s, \xi) \cdot \xi \geq a_1 |\xi|^{p(x)} - a_2 |s|^{q_1(x)} - a_3$ for a.a. $x \in \Omega$;

(ii) $|A(x, s, \xi)| \leq a_4 |\xi|^{p(x)-1} + a_5 |s|^{q_1(x) \frac{p(x)-1}{p(x)}} + a_6$ for a.a. $x \in \Omega$;

(iii) $|F(x,s,\xi)| \le b_1|\xi|^{p(x)\frac{q_1(x)-1}{q_1(x)}} + b_2|s|^{q_1(x)-1} + b_3$ for a.a. $x \in \Omega$;
(iv) $|\eta| \le c_1|s|^{q_1(x)-1} + c_2$ for a.a. $x \in \Omega$, for all $\eta \in \partial j_1(x,s)$;
(v) $|\tau| \le d_1|s|^{q_2(x)-1} + d_2$, for a.a. $x \in \partial\Omega$, for all $\tau \in \partial j_2(x,s)$;

and for all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^N$ with positive constants a_i, b_j, c_k, d_l ($i \in \{1,\dots,6\}$, $j \in \{1,\dots,3\}$, $k,l \in \{1,2\}$) and fixed functions $p, q_1 \in C(\overline{\Omega})$, $q_2 \in C(\partial\Omega)$ such that $1 < \inf_{x\in\overline{\Omega}} p(x)$ and

$$p(x) \le q_1(x) < p^*(x),\ x \in \overline{\Omega}, \quad p(x) \le q_2(x) < p_*(x),\ x \in \partial\Omega, \tag{4.1}$$

with the critical exponents

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N, \end{cases} \quad p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

As in the case of constant exponents we obtain a certain growth rate of the generalized directional derivatives of the form

$$|j_1^\circ(x,s;r)| \le \left(c_1|s|^{q_1(x)-1} + c_2\right)|r| \quad \text{for a.a. } x \in \Omega \text{ and for all } s,r \in \mathbb{R}, \tag{4.2}$$

$$|j_2^\circ(x,s;r)| \le \left(d_1|s|^{q_2(x)-1} + d_2\right)|r| \quad \text{for a.a. } x \in \partial\Omega \text{ and for all } s,r \in \mathbb{R}. \tag{4.3}$$

Then, the finiteness of the left-hand side in (1.1) is a consequence of the compact embedding $W^{1,p(\cdot)}(\Omega) \to L^{q_1(\cdot)}(\Omega)$ and the fact that the trace operator is a bounded operator from $W^{1,p(\cdot)}(\Omega)$ into $L^{q_2(\cdot)}(\partial\Omega)$ (see Fan-Shen-Zhao [14, Theorem 1.3] and Fan [13, Corollary 2.4]). Notice that we do not need log-Hölder continuity conditions, the variable exponents p, q_1, and q_2 are only supposed to be continuous.

We assume the following hypotheses on the set K.

(K3) For $u \in K$ and $\kappa > 0$ there exists $t > 0$ such that

$$\varphi = u - t(u-\kappa)_+ \in K.$$

(K4) For $u \in K$ and $\kappa > 0$ there exists $h > 0$ such that

$$\psi = u - h(u+\kappa)_- \in K.$$

Since $W^{1,p(\cdot)}(\Omega)$ has lattice structure we notice that both function φ and ψ belong to $W^{1,p(\cdot)}(\Omega)$.

We start with the result on truncated energy estimates.

Proposition 4.1 *Assume hypothesis (H2) and let $u \in K$ be a solution of (1.1).*

(1) *If condition (K3) is satisfied, then*

$$\int_{A_\kappa} |\nabla u|^{p(x)} dx \le \hat{M}_1 \int_{A_\kappa} u^{q_1(x)} dx + \hat{M}_2 \int_{\partial A_\kappa} u^{q_2(x)} d\sigma,$$

where

$$A_\kappa = \{x \in \Omega : u(x) > \kappa\}, \qquad \partial A_\kappa = \{x \in \partial\Omega : u(x) > \kappa\}, \quad \kappa \ge 1,$$

and with positive constants $\hat{M}_1 = \hat{M}_1(q_1, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2)$ and $\hat{M}_2 = \hat{M}_2(a_1, d_1, d_2)$.

(2) *If condition (K4) is satisfied, then*

$$\int_{\tilde{A}_\kappa} |\nabla u|^{p(x)} dx \le \hat{M}_1 \int_{\tilde{A}_\kappa} (-u)^{q_1(x)} dx + \hat{M}_2 \int_{\partial \tilde{A}_\kappa} (-u)^{q_2(x)} d\sigma,$$

where

$$\tilde{A}_\kappa = \{x \in \Omega : -u(x) > \kappa\}, \qquad \partial \tilde{A}_\kappa = \{x \in \partial\Omega : -u(x) > \kappa\}, \quad \kappa \ge 1,$$

with the same constants $\hat{M}_1$ and $\hat{M}_2$ as in part (1).

Proof Let $\kappa \ge 1$ be fixed and let $u \in K$ be a solution of (1.1). Due to (K3) we may choose $v = \varphi = u - t(u-\kappa)_+ \in K$ in (1.1) to get

$$\begin{aligned} &\int_{A_\kappa} A(x,u,\nabla u) \cdot \nabla (t(u-\kappa)) dx \\ &\le \int_{A_\kappa} F(x,u,\nabla u)(-t(u-\kappa)) dx + \int_{A_\kappa} j_1^\circ(x,u;-t(u-\kappa)) dx \\ &\quad + \int_{\partial A_\kappa} j_2^\circ(x,u;-t(u-\kappa)) d\sigma. \end{aligned} \tag{4.4}$$

Since $r \mapsto j_k^\circ(x,s;r)$ is positively homogeneous (see [9, Proposition 2.1.1]) we may divide inequality (4.4) by $t > 0$. By virtue of condition (H2)(i) we obtain for the left-hand side of (4.4)

$$\begin{aligned} &\int_{A_\kappa} A(x,u,\nabla u) \cdot \nabla (u-\kappa) dx \\ &= \int_{A_\kappa} A(x,u,\nabla u) \cdot \nabla u dx \\ &\ge \int_{A_\kappa} \left(a_1 |\nabla u|^{p(x)} - a_2 |u|^{q_1(x)} - a_3 \right) dx \\ &\ge a_1 \int_{A_\kappa} |\nabla u|^{p(x)} dx - (a_2 + a_3) \int_{A_\kappa} |u|^{q_1(x)} dx, \end{aligned} \tag{4.5}$$

where $u^{q_1(x)} > u > 1$ in A_κ was taken into account. The first term on the right-hand side can be estimated via Young's inequality with $\varepsilon \in (0,1]$ and condition (H2)(iii). This leads to

$$\begin{aligned} &\int_{A_\kappa} F(x,u,\nabla u)(-(u-\kappa)) dx \\ &\le \int_{A_\kappa} \left[b_1 |\nabla u|^{p(x)\frac{q_1(x)-1}{q_1(x)}} + b_2 |u|^{q_1(x)-1} + b_3 \right] (u-\kappa) dx \\ &\le b_1 \int_{A_\kappa} \left[\varepsilon^{\frac{q_1(x)-1}{q_1(x)}} |\nabla u|^{p(x)\frac{q_1(x)-1}{q_1(x)}} \varepsilon^{-\frac{q_1(x)-1}{q_1(x)}} u \right] dx + (b_2+b_3) \int_{A_\kappa} u^{q_1(x)} dx \\ &\le b_1 \int_{A_\kappa} \varepsilon |\nabla u|^{p(x)} dx + b_1 \int_{A_\kappa} \varepsilon^{-(q_1(x)-1)} u^{q_1(x)} dx \\ &\quad + (b_2+b_3) \int_{A_\kappa} u^{q_1(x)} dx \\ &\le \varepsilon b_1 \int_{A_\kappa} |\nabla u|^{p(x)} dx + \left(b_1 \varepsilon^{-(q_1^+ - 1)} + b_2 + b_3 \right) \int_{A_\kappa} u^{q_1(x)} dx, \end{aligned} \tag{4.6}$$

where $q_1^+ = \max_{\overline{\Omega}} q_1$. By means of hypothesis (H2)(iv) (see also (4.2)) the second term on the right-hand side of (4.4) gives

$$\begin{aligned}\int_{A_\kappa} j_1^\circ(x,u;-(u-\kappa))dx &\leq \int_{A_\kappa} \left(c_1|u|^{q_1(x)-1}+c_2\right)(u-\kappa)dx \\ &\leq (c_1+c_2)\int_{A_\kappa} u^{q_1(x)}dx.\end{aligned} \tag{4.7}$$

Finally, due to (H2)(v) (see also (4.3)), we arrive

$$\begin{aligned}\int_{\partial A_\kappa} j_2^\circ(x,u;-(u-\kappa))d\sigma &\leq \int_{\partial A_\kappa} \left(d_1|u|^{q_2(x)}+d_2\right)(u-\kappa)d\sigma \\ &\leq (d_1+d_2)\int_{\partial A_\kappa} u^{q_2(x)}d\sigma.\end{aligned} \tag{4.8}$$

Combining (4.4)–(4.8) and choosing $\varepsilon = \min\left(1, \frac{a_1}{2b_1}\right)$ yields

$$\begin{aligned}&\frac{a_1}{2}\int_{A_\kappa} |\nabla u|^{p(x)}dx \\ &\leq \left(a_2+a_3+b_1\varepsilon^{-(q_1^+-1)}+b_2+b_3+c_1+c_2\right)\int_{A_\kappa} u^{q_1(x)}dx \\ &\quad + (d_1+d_2)\int_{\partial A_\kappa} u^{q_2(x)}d\sigma.\end{aligned} \tag{4.9}$$

Now we may divide (4.9) by $\frac{a_1}{2} > 0$ which yields claim (1).

In order to prove part (2) we take $v = \psi = u - h(u+\kappa)_- \in K$ as test function in (1.1). This leads to

$$\begin{aligned}&h\int_{\tilde{A}_\kappa} A(x,u,\nabla u)\cdot\nabla(u+\kappa)dx \\ &\leq -h\int_{\tilde{A}_\kappa} F(x,u,\nabla u)(u+\kappa)dx + h\int_{\tilde{A}_\kappa} j_1^\circ(x,u;-(u+\kappa))dx \\ &\quad + h\int_{\partial\tilde{A}_\kappa} j_2^\circ(x,u;-(u+\kappa))d\sigma.\end{aligned}$$

Dividing again by $h > 0$ and applying the structure conditions in (H2)(i), (iii)–(v) we obtain the estimate in (2). □

Now we can state our main result on upper and lower bounds for solutions of (1.1).

Theorem 4.2 *Assume hypothesis (H2) and let $u \in K$ be a solution of (1.1). Then there exists a constant $C_2 > 0$ such that the following assertions hold.*

(1) *If condition (K3) is satisfied, then*

$$\operatorname*{ess\,sup}_{x\in\Omega} u(x) \leq C_2.$$

(2) *If condition (K4) is satisfied, then*

$$\operatorname*{ess\,inf}_{x\in\Omega} u(x) \geq -C_2.$$

Proof The proof can exactly be done as in [32, Theorem 1.1] where Lemma 3.1 and 3.2 in [32] have to be replaced by Proposition 4.1. □

Analogous to Section 3 we have the corresponding result on the boundedness of solutions of (1.1).

Corollary 4.3 *Let the conditions in (H2), (K3), and (K4) be satisfied and let $u \in K$ be a solution of* (1.1)*. Then there holds*

$$\operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \le C_2,$$

with the same constant C_2 stated in Theorem 4.2s.

Remark 4.4 As proved in [32] the constant C_2 given in Theorem 4.2 and Corollary 4.3 can be characterized through

$$C_2 = 2\max\left(1, \hat{M}_3\left[\int_\Omega u_+^{q_1(x)}dx + \int_{\partial\Omega} u_+^{q_2(x)}d\sigma\right]^{\hat{M}_4}\right)$$

with positive constants $\hat{M}_3 = \hat{M}_3(p, q_1, q_2, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, d_1, d_2, N, \Omega)$ and $\hat{M}_4 = \hat{M}_4(p, q_1, q_2)$. As already mentioned in the constant exponent case the growth condition for $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ (see (H2)(ii)) is not needed in the proof of Theorem 4.2.

Remark 4.5 The proof of Theorem 1.1 in [32] is mainly based on the localization method combined with an appropriate choice of the partition of unity. Since $\overline{\Omega}$ is compact we find, for any $R > 0$, a finite open cover $\{B_i(R)\}_{i=1,\dots,m}$ of balls $B_i := B_i(R)$ with radius R such that $\overline{\Omega} \subset \bigcup_{i=1}^m B_i(R)$. Because of the continuity of p, q_1, q_2 and due to (4.1) we may take $R > 0$ small enough such that

$$p_i^+ \le q_{1,i}^+ < (p_i^-)^*, \qquad p_i^+ \le q_{2,i}^+ < (p_i^-)_*, \qquad i = 1, \dots, m,$$

where

$$p_i^+ = \max_{x\in B_i\cap\overline{\Omega}} p(x), \quad q_{1,i}^+ = \max_{x\in B_i\cap\overline{\Omega}} q_1(x),$$
$$p_i^- = \min_{x\in B_i\cap\overline{\Omega}} p(x), \quad q_{2,i}^+ = \max_{x\in B_i\cap\partial\Omega} q_2(x),$$

and $(p_i^-)^*$, $(p_i^-)_*$ denote the usual critical exponents of p_i^-. Then, we can choose a partition of unity $\{\xi_i\}_{i=1}^m \subset C_0^\infty(\mathbb{R}^N)$ with respect to the open cover $\{B_i(R)\}_{i=1,\dots,m}$. That means, we have

$$\operatorname{supp}\xi_i \subset B_i, \quad 0 \le \xi_i \le 1, \quad i = 1, \dots, m, \quad \text{and} \quad \sum_{i=1}^m \xi_i = 1 \ \text{ on } \overline{\Omega}.$$

In fact, the idea is to treat problems involving nonstandard growth conditions as problems with constant exponent growth rates. It arises the open question what happens if p is not continuous, but essentially bounded. In this case the ideas above cannot be applied.

Remark 4.6 In contrast to De Giorgi's iteration it seems that Moser's iteration is less suitable for problems with variable exponents concerning global a priori bounds. With view to the assumptions in (K1) and (K2), we see that p occurs as exponent in the test function. This makes the corresponding gradient more complicated and it is not clear for the author if this method works in the variable exponent case even if $K = W^{1,p(\cdot)}(\Omega)$ and j_k $(k = 1, 2)$ smooth.

Now we give sufficient conditions for the set K satisfying the conditions in (K3) and (K4).

Lemma 4.7

(1) *If K contains the positive constant functions and if K satisfies the condition $K \wedge K \subset K$, then the hypothesis (K3) is satisfied with $t = 1$.*
(2) *If K contains the negative constant functions and if K satisfies the condition $K \vee K \subset K$, then the hypothesis (K4) is satisfied with $h = 1$.*
(3) *If K contains the constant functions and if K has lattice structure, then the hypotheses (K3) and (K4) are satisfied with $t = h = 1$.*

Proof Let $u \in K$. Since K is closed under "$\wedge$" we obtain, for $\kappa \in K$ with $\kappa > 0$,

$$\min(u, \kappa) = u - (u - \kappa)_+ \in K.$$

Hence, condition (K3) is satisfied for $t = 1$. This proves (1). Similarly, we have, for $u \in K$ and $-\kappa \in K$ with $\kappa > 0$,

$$\max(u, -\kappa) = u - (u + \kappa)_- \in K,$$

as K is closed under "$\vee$". Then, condition (K4) is fulfilled for $h = 1$ which yields the assertion in (2). The last part follows directly from (1) and (2). □

Let us consider some examples.

Example 4.8 Let $\vartheta : \Omega \to \mathbb{R}$ be a given function with $\vartheta \geq 0$ a.e. in Ω and consider the sets

$$K = \{v \in W^{1,p(\cdot)}(\Omega) : v \leq \vartheta \quad \text{a.e. in } \Omega\},$$
$$K = \{v \in W^{1,p(\cdot)}(\Omega) : v \geq -\vartheta \quad \text{a.e. in } \Omega\},$$
$$K = \{v \in W^{1,p(\cdot)}(\Omega) : |\nabla v| \leq \vartheta \quad \text{a.e. in } \Omega\}.$$

We observe that these sets have lattice structure, that means, they fulfill (2.5). Moreover, the first set contains the negative constant functions, the second set the positive constant functions and the last one the constant functions. Hence, the assumptions of Lemma 4.7 are satisfied and Theorem 4.2 can be applied to these sets.

As a special case of the second set we have the cone of nonnegative elements of $W^{1,p(\cdot)}(\Omega)$, that is

$$K_{\geq 0} = \{v \in W^{1,p(\cdot)}(\Omega) : v \geq 0 \quad \text{a.e. in } \Omega\}.$$

It is clear that $K_{\geq 0}$ has lattice structure and contains the positive constant functions. Hence, Lemma 4.7(1) implies that $K_{\geq 0}$ satisfies (K3) and from Theorem 4.2(1) we infer that every solution $u \in K_{\geq 0}$ of (1.1) is bounded from above. Since $K_{\geq 0}$ is bounded from below by zero, we have that $u \in L^\infty(\Omega)$.

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On the Fučik spectrum for the p-Laplacian with Robin boundary condition

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ABSTRACT

The aim of this paper is to study the Fučik spectrum of the p-Laplacian with Robin boundary condition given by

$$\begin{aligned} -\Delta_p u &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= -\beta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned}$$

where $\beta \geq 0$. If $\beta = 0$, it reduces to the Fučik spectrum of the negative Neumann p-Laplacian. The existence of a first nontrivial curve $\mathcal{C}$ of this spectrum is shown and we prove some properties of this curve, e.g., $\mathcal{C}$ is Lipschitz continuous, decreasing and has a certain asymptotic behavior. A variational characterization of the second eigenvalue λ_2 of the Robin eigenvalue problem involving the p-Laplacian is also obtained.

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1. Introduction

The Fučik spectrum of the negative p-Laplacian with a Robin boundary condition is defined as the set $\widehat{\Sigma}_p$ of $(a, b) \in \mathbb{R}^2$ such that

$$\begin{aligned} -\Delta_p u &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= -\beta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

has a nontrivial solution. Here the domain $\Omega \subset \mathbb{R}^N$ is supposed to be bounded with a smooth boundary $\partial\Omega$. The notation $-\Delta_p u$ stands for the negative p-Laplacian of u, i.e., $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, with $1 < p < +\infty$, while $\frac{\partial u}{\partial \nu}$ denotes the outer normal derivative of u and β is a parameter belonging to $[0, +\infty)$. We also denote $u^{\pm} = \max\{\pm u, 0\}$. For $\beta = 0$, (1.1) becomes the Fučik spectrum of the negative Neumann p-Laplacian. Let us recall that $u \in W^{1,p}(\Omega)$ is a (weak) solution of (1.1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \mathrm{d}x + \beta \int_{\partial\Omega} |u|^{p-2}uv \mathrm{d}\sigma = \int_\Omega (a(u^+)^{p-1} - b(u^-)^{p-1})v \mathrm{d}x, \quad \forall v \in W^{1,p}(\Omega). \tag{1.2}$$

If $a = b = \lambda$, problem (1.1) reduces to

$$\begin{aligned} -\Delta_p u &= \lambda|u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= -\beta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

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which is known as the Robin eigenvalue problem for the p-Laplacian. As proved in [1], the first eigenvalue λ_1 of problem (1.3) is simple, isolated and can be characterized as follows

$$\lambda_1 = \inf_{u\in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p \mathrm{d}x + \beta \int_{\partial\Omega} |u|^p \mathrm{d}\sigma \,:\, \int_\Omega |u|^p \mathrm{d}x = 1 \right\}.$$

The author also proves that the eigenfunctions corresponding to λ_1 are of constant sign and belong to $C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Throughout this paper, φ_1 denotes the eigenfunction of (1.3) associated to λ_1 which is normalized as $\|\varphi_1\|_{L^p(\Omega)} = 1$ and satisfies $\varphi_1 > 0$. Let us also recall that every eigenfunction of (1.3) corresponding to an eigenvalue $\lambda > \lambda_1$ must change sign.

We briefly describe the context of the Fučik spectrum related to problem (1.1). The Fučik spectrum was introduced by Fučik [2] in the case of the negative Laplacian in one dimension with periodic boundary conditions. He proved that this spectrum is composed of two families of curves emanating from the points (λ_k, λ_k) determined by the eigenvalues λ_k of the problem. Afterwards, many authors studied the Fučik spectrum Σ_2 for the negative Laplacian with Dirichlet boundary conditions (see [3–11] and the references therein). In this respect, we mention that Dancer [12] proved that the lines $\mathbb{R}\times\{\lambda_1\}$ and $\{\lambda_1\}\times\mathbb{R}$ are isolated in Σ_2. De Figueiredo–Gossez [13] constructed a first nontrivial curve in Σ_2 through (λ_2, λ_2) and characterized it variationally. For $p \neq 2$ and in one dimension, Drábek [14] has shown that Σ_p has similar properties as in the linear case, i.e., $p = 2$. The Fučik spectrum Σ_p of the negative p-Laplacian with homogeneous Dirichlet boundary conditions in the general case $1 < p < +\infty$ and $N \geq 1$, that is

$$(a,b) \in \Sigma_p : \begin{array}{rcll} -\Delta_p u &=& a(u^+)^{p-1} - b(u^-)^{p-1} & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array}$$

has been studied by Cuesta et al. [15], where the authors proved the existence of a first nontrivial curve through (λ_2, λ_2) and that the lines $\mathbb{R}\times\{\lambda_1\}$ and $\{\lambda_1\}\times\mathbb{R}$ are isolated in Σ_p. For other results on Σ_p we refer to [16–20].

The Fučik spectrum Θ_p of the negative p-Laplacian with homogeneous Neumann boundary condition, which is defined by

$$(a,b) \in \Theta_p : \begin{array}{rcll} -\Delta_p u &=& a(u^+)^{p-1} - b(u^-)^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} &=& 0 & \text{on } \partial\Omega, \end{array}$$

was investigated in [21–23]. It is worth emphasizing that Arias et al. [22] pointed out an important difference between the cases $p \leq N$ and $p > N$ regarding the asymptotic properties of the first nontrivial curve in Θ_p. Note that the Fučik spectrum Θ_p is incorporated in problem (1.1) by taking $\beta = 0$. Finally, we mention the work of Martínez and Rossi [24] who considered the Fučik spectrum $\widetilde{\Sigma}_p$ associated to Steklov boundary condition, which is introduced by

$$(a,b) \in \widetilde{\Sigma}_p : \begin{array}{rcll} -\Delta_p u &=& -|u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} &=& a(u^+)^{p-1} - b(u^-)^{p-1} & \text{on } \partial\Omega. \end{array}$$

As in the previous situations, they constructed a first nontrivial curve in $\widetilde{\Sigma}_p$ through (λ_2, λ_2), where λ_2 denotes the second eigenvalue of the Steklov eigenvalue problem, and studied its asymptotic behavior.

The aim of this paper is the study of the Fučik spectrum $\widehat{\Sigma}_p$ given in (1.1) for the negative p-Laplacian with Robin boundary condition. We are going to prove the existence of a first nontrivial curve $\mathcal{C}$ of this spectrum and show that it shares the same properties as in the cases of the other problems discussed above: Lipschitz continuity, strictly decreasing monotonicity and asymptotic behavior. It is a significant fact that the presence of the parameter β in problem (1.1) does not alter these basic properties. The main idea in studying the asymptotic behavior of the curve $\mathcal{C}$ is the use of a suitable equivalent norm related to β. A relevant consequence of the construction of the first nontrivial curve $\mathcal{C}$ in $\widehat{\Sigma}_p$ is the following variational characterization of the second eigenvalue λ_2 of (1.3):

$$\lambda_2 = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} \left[\int_\Omega |\nabla u|^p \mathrm{d}x + \beta \int_{\partial\Omega} |u|^p \mathrm{d}\sigma \right], \tag{1.4}$$

where

$$\Gamma = \{\gamma \in C([-1,1], S) : \gamma(-1) = -\varphi_1,\ \gamma(1) = \varphi_1\},$$

with

$$S = \left\{ u \in W^{1,p}(\Omega) : \int_\Omega |u|^p \mathrm{d}x = 1 \right\}. \tag{1.5}$$

The results presented in this paper complete the picture of the Fučik spectrum involving the p-Laplacian by adding in the case of Robin condition the information previously known for Dirichlet problem (see [15]), Steklov problem (see [24]),

and homogeneous Neumann problem (see [22]). Actually, as already specified, the results given here for the Fučik spectrum (1.1) of the negative p-Laplacian with Robin boundary condition extend the ones known for the Fučik spectrum Θ_p under Neumann boundary condition by simply making $\beta = 0$.

Our approach is variational relying on the functional associated to problem (1.1), which is expressed on $W^{1,p}(\Omega)$ by

$$J(u) = \int_\Omega |\nabla u|^p \mathrm{d}x + \beta \int_{\partial\Omega} |u|^p \mathrm{d}\sigma - \int_\Omega (a(u^+)^p + b(u^-)^p)\mathrm{d}x.$$

It is clear that $J \in C^1(W^{1,p}(\Omega), \mathbb{R})$ and the critical points of J coincide with the weak solutions of problem (1.1). In comparison with the corresponding functionals related to the Fučik spectrum for the Dirichlet and Steklov problems, the functional J exhibits an essential difference because its expression does not contain the norm of the space $W^{1,p}(\Omega)$, and it is also different from the functional used to treat the Neumann problem because it contains the additional boundary term involving β. However, in our proofs various ideas and techniques are worked out on the pattern of [22,15,24].

The rest of the paper is organized as follows. Section 2 is devoted to the determination of elements of $\widehat{\Sigma}_p$ by means of critical points of a suitable functional. Section 3 sets forth the construction of the first nontrivial curve $\mathcal{C}$ in $\widehat{\Sigma}_p$ and the variational characterization of the second eigenvalue λ_2 for (1.3). Section 4 presents the basic properties of $\mathcal{C}$.

2. The spectrum $\widehat{\Sigma}_p$ through critical points

The aim of this section is to determine elements of the Fučik spectrum $\widehat{\Sigma}_p$ defined in problem (1.1). They are found by critical points of a functional that is constructed by means of the Robin problem (1.1). To this end we follow certain ideas in [15,24] developed for problems with Dirichlet and Steklov boundary conditions.

For a fixed $s \in \mathbb{R}$, $s \geq 0$, and corresponding to $\beta \geq 0$ given in problem (1.1), we introduce the functional $J_s : W^{1,p}(\Omega) \to \mathbb{R}$ by

$$J_s(u) = \int_\Omega |\nabla u|^p \mathrm{d}x + \beta \int_{\partial\Omega} |u|^p \mathrm{d}\sigma - s \int_\Omega (u^+)^p \mathrm{d}x,$$

thus $J_s \in C^1(W^{1,p}(\Omega), \mathbb{R})$. The set S introduced in (1.5) is a smooth submanifold of $W^{1,p}(\Omega)$, and thus $\widetilde{J}_s = J_s|_S$ is a C^1 function in the sense of manifolds. We note that $u \in S$ is a critical point of $\widetilde{J}_s$ (in the sense of manifolds) if and only if there exists $t \in \mathbb{R}$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \mathrm{d}x + \beta \int_{\partial\Omega} |u|^{p-2}uv \mathrm{d}\sigma - s \int_\Omega (u^+)^{p-1} v \mathrm{d}x = t \int_\Omega |u|^{p-2}uv \mathrm{d}x, \quad \forall v \in W^{1,p}(\Omega). \tag{2.1}$$

Now we describe the relationship between the critical points of $\widetilde{J}_s$ and the spectrum $\widehat{\Sigma}_p$.

Lemma 2.1. *Given a number $s \geq 0$, one has that $(s + t, t) \in \mathbb{R}^2$ belongs to the spectrum $\widehat{\Sigma}_p$ if and only if there exists a critical point $u \in S$ of $\widetilde{J}_s$ such that $t = J_s(u)$.*

Proof. The definition in (1.2) for the weak solution shows that $(t + s, t) \in \widehat{\Sigma}_p$ if and only if there is $u \in S$ that solves the Robin problem

$$\begin{aligned} -\Delta_p u &= (t+s)(u^+)^{p-1} - t(u^-)^{p-1} && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= -\beta |u|^{p-2}u && \text{on } \partial\Omega, \end{aligned}$$

which means exactly (2.1). Inserting $v = u$ in (2.1) yields $t = J_s(u)$, as required. □

Lemma 2.1 enables us to find points in $\widehat{\Sigma}_p$ through the critical points of $\widetilde{J}_s$. In order to implement this, first we look for minimizers of $\widetilde{J}_s$.

Proposition 2.2. *There hold:*
(i) *the first eigenfunction φ_1 is a global minimizer of $\widetilde{J}_s$;*
(ii) *the point $(\lambda_1, \lambda_1 - s) \in \mathbb{R}^2$ belongs to $\widehat{\Sigma}_p$.*

Proof. (i) Since $\beta, s \geq 0$, using the characterization of λ_1 we have

$$\widetilde{J}_s(u) = \int_\Omega |\nabla u|^p \mathrm{d}x + \beta \int_{\partial\Omega} |u|^p \mathrm{d}\sigma - s \int_\Omega (u^+)^p \mathrm{d}x \geq \lambda_1 \int_\Omega |u|^p \mathrm{d}x - s \int_\Omega (u^+)^p \mathrm{d}x \geq \lambda_1 - s = J_s(\varphi_1),$$
$$\forall u \in S.$$

(ii) On the basis of (i), we can apply Lemma 2.1. □

Next we produce a second critical point of $\widetilde{J}_s$ as a local minimizer.

Proposition 2.3. *There hold:*
(i) *the negative eigenfunction $-\varphi_1$ is a strict local minimizer of $\widetilde{J}_s$;*
(ii) *the point $(\lambda_1 + s, \lambda_1) \in \mathbb{R}^2$ belongs to $\widehat{\Sigma}_p$.*

Proof. (i) Arguing indirectly, let us suppose that there exists a sequence $(u_n) \subset S$ with $u_n \neq -\varphi_1$, $u_n \to -\varphi_1$ in $W^{1,p}(\Omega)$ and $J_s(u_n) \leq \lambda_1 = J_s(-\varphi_1)$. If $u_n \leq 0$ for a.a. $x \in \Omega$, we obtain

$$\widetilde{J}_s(u_n) = \int_\Omega |\nabla u_n|^p \mathrm{d}x + \beta \int_{\partial\Omega} |u_n|^p \mathrm{d}\sigma > \lambda_1,$$

because $u_n \neq -\varphi_1$ and $u_n \neq \varphi_1$, which contradicts the assumption $\widetilde{J}_s(u_n) \leq \lambda_1$.

Consider now the complementary situation. Hence u_n changes sign whenever n is sufficiently large, thereby we can set

$$w_n = \frac{u_n^+}{\|u_n^+\|_{L^p(\Omega)}} \quad \text{and} \quad r_n = \|\nabla w_n\|_{L^p(\Omega)}^p + \beta \|w_n\|_{L^p(\partial\Omega)}^p. \tag{2.2}$$

We claim that, along a relabeled subsequence, $r_n \to +\infty$ as $n \to \infty$. Suppose by contradiction that (r_n) is bounded. This implies through (2.2) that (w_n) is bounded in $W^{1,p}(\Omega)$, so there exists a subsequence denoted again by (w_n) such that $w_n \to w$ in $L^p(\Omega)$, for some $w \in W^{1,p}(\Omega)$. Since $\|w_n\|_{L^p(\Omega)} = 1$ and $w_n \geq 0$ a.e. in Ω, we get $\|w\|_{L^p(\Omega)} = 1$ and $w \geq 0$, hence the measure of the set $\{x \in \Omega : u_n(x) > 0\}$ does not approach 0 when $n \to \infty$. This contradicts the assumption that $u_n \to -\varphi_1$ in $L^p(\Omega)$, thus proving the claim.

On the other hand, from (2.2) and by using the variational characterization of λ_1, we infer that

$$\begin{aligned}\widetilde{J}_s(u_n) &= (r_n - s) \int_\Omega |u_n^+|^p \mathrm{d}x + \int_\Omega |\nabla u_n^-|^p \mathrm{d}x + \beta \int_{\partial\Omega} |u_n^-|^p \mathrm{d}\sigma \\ &\geq (r_n - s) \int_\Omega |u_n^+|^p \mathrm{d}x + \lambda_1 \int_\Omega (u_n^-)^p \mathrm{d}x,\end{aligned}$$

whereas the choice of (u_n) gives

$$\widetilde{J}_s(u_n) \leq \lambda_1 = \lambda_1 \int_\Omega (u_n^+)^p \mathrm{d}x + \lambda_1 \int_\Omega (u_n^-)^p \mathrm{d}x.$$

Combining the inequalities above results in

$$(\lambda_1 - r_n + s) \int_\Omega (u_n^+)^p \mathrm{d}x \geq 0,$$

therefore $\lambda_1 \geq r_n - s$. This is against the unboundedness of (r_n), which completes the proof of (i). Part (ii) follows from Lemma 2.1 because $J_s(-\varphi_1) = \lambda_1$. □

Using the two local minima obtained in Propositions 2.2 and 2.3, we seek for a third critical point of $\widetilde{J}_s$ via a version of the Mountain-Pass theorem on C^1-manifolds.

We define a norm of the derivative of the restriction $\widetilde{J}_s$ of J_s to S at the point $u \in S$ by

$$\|\widetilde{J}_s'(u)\|_* = \min\{\|J_s'(u) - tT'(u)\|_{(W^{1,p}(\Omega))^*} : t \in \mathbb{R}\},$$

where $T(\cdot) = \|\cdot\|_{L^p(\Omega)}^p$. Let us recall the definition of the Palais–Smale condition.

Definition 2.4. The functional $J_s : W^{1,p}(\Omega) \to \mathbb{R}$ is said to satisfy the Palais–Smale condition on S if for any sequence $(u_n) \subset S$ such that $(J_s(u_n))$ is bounded and $\|\widetilde{J}_s'(u_n)\|_* \to 0$ as $n \to \infty$, there exists a strongly convergent subsequence in $W^{1,p}(\Omega)$.

Note that the negative p-Laplacian $-\Delta_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ fulfills the $(S)_+$-property, that means, if

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega) \text{ and } \limsup_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla(u_n - u) \leq 0,$$

then it holds $u_n \to u$ in $W^{1,p}(\Omega)$ (cf. [25,26]). Taking into account this property, we first check the Palais–Smale condition for $\widetilde{J}_s$ on the submanifold S of $W^{1,p}(\Omega)$.

Lemma 2.5. *The functional $\widetilde{J}_s : S \to \mathbb{R}$ satisfies the Palais–Smale condition on S in the sense of manifolds.*

Proof. Let $(u_n) \subset S$ be a sequence provided $(J_s(u_n))$ is bounded and $\|\widetilde{J}_s'(u_n)\|_* \to 0$ as $n \to \infty$, which means that there exists a sequence $(t_n) \subset \mathbb{R}$ such that

$$\left| \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v \mathrm{d}x + \beta \int_{\partial\Omega} |u_n|^{p-2} u_n v \mathrm{d}\sigma - s \int_\Omega (u_n^+)^{p-1} v \mathrm{d}x - t_n \int_\Omega |u_n|^{p-2} u_n v \mathrm{d}x \right| \leq \varepsilon_n \|v\|_{W^{1,p}(\Omega)}, \tag{2.3}$$

for all $v \in W^{1,p}(\Omega)$ and with $\varepsilon_n \to 0^+$. Note that $J_s(u_n) \geq \|\nabla u_n\|_{L^p(\Omega)}^p - s$. Since $(u_n) \in S$ and $(J_s(u_n))$ is bounded, we derive that (u_n) is bounded in $W^{1,p}(\Omega)$. Thus, along a relabeled subsequence we may suppose that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, $u_n \to u$

in $L^p(\Omega)$ and $u_n \to u$ in $L^p(\partial\Omega)$. Taking $v = u_n$ in (2.3) and using again $(u_n) \subset S$ show that the sequence (t_n) is bounded. Then, if we choose $v = u_n - u$, it follows that

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u)dx \to 0 \quad \text{as } n \to \infty.$$

At this point, the $(S)_+$-property of $-\Delta_p$ on $W^{1,p}(\Omega)$ enables us to conclude that $u_n \to u$ in $W^{1,p}(\Omega)$. □

The proof of the following version of the Mountain-Pass theorem can be found in [27, Theorem 3.2]

Theorem 2.6. *Let E be a Banach space and let $g, f \in C^1(E, \mathbb{R})$. Further, suppose that 0 is a regular value of g and let $M = \{u \in E : g(u) = 0\}$, $u_0, u_1 \in M$ and $\varepsilon > 0$ such that $\|u_1 - u_0\|_E > \varepsilon$ and*

$$\inf\{f(u) : u \in M \text{ and } \|u - u_0\|_E = \varepsilon\} > \max\{f(u_0), f(u_1)\}.$$

Assume that f satisfies the Palais–Smale condition on M and that

$$\Gamma = \{\gamma \in C([-1, 1], M) : \gamma(-1) = u_0 \text{ and } \gamma(1) = u_1\}$$

is nonempty. Then

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} f(u)$$

is a critical value of $f|_M$.

Now we obtain, in addition to φ_1 and $-\varphi_1$, a third critical point of $\widetilde{J}_s$ on S.

Proposition 2.7. *There hold that, for each $s \geq 0$:*

(i)

$$c(s) := \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} J_s(u), \tag{2.4}$$

where

$$\Gamma = \{\gamma \in C([-1, 1], S) : \gamma(-1) = -\varphi_1 \text{ and } \gamma(1) = \varphi_1\},$$

is a critical value of $\widetilde{J}_s$ satisfying $c(s) > \max\{\widetilde{J}_s(-\varphi_1), \widetilde{J}_s(\varphi_1)\} = \lambda_1$. In particular, there exists a critical point of $\widetilde{J}_s$ that is different from $-\varphi_1$ and φ_1.

(ii) *The point $(s + c(s), c(s))$ belongs to $\widehat{\Sigma}_p$.*

Proof. (i) By Proposition 2.3 we know that $-\varphi_1$ is a strict local minimizer of $\widetilde{J}_s$ with $\widetilde{J}_s(-\varphi_1) = \lambda_1$, while Proposition 2.2 ensures that φ_1 is a global minimizer of $\widetilde{J}_s$ with $J_s(\varphi_1) = \lambda_1 - s$. Then we can show that

$$\inf\{\widetilde{J}_s(u) : u \in S \text{ and } \|u - (-\varphi_1)\|_{W^{1,p}(\Omega)} = \varepsilon\} > \max\{\widetilde{J}_s(-\varphi_1), \widetilde{J}_s(\varphi_1)\} = \lambda_1, \tag{2.5}$$

whenever $\varepsilon > 0$ is sufficiently small. The proof that the inequality above is strict can be done as in [15, Lemma 2.9] (see also [24, Lemma 2.6]) on the basis of Ekeland's variational principle. In order to fulfill the Mountain-Pass geometry we choose $\varepsilon > 0$ even smaller if necessary to have $2\|\varphi_1\|_{W^{1,p}(\Omega)} = \|\varphi_1 - (-\varphi_1)\|_{W^{1,p}(\Omega)} > \varepsilon$. Since $\widetilde{J}_s : S \to \mathbb{R}$ satisfies the Palais–Smale condition on the manifold S as shown in Lemma 2.5, we may invoke the version of Mountain-Pass theorem on manifolds in Theorem 2.6. This guarantees that $c(s)$ introduced in (2.4) is a critical value of $\widetilde{J}_s$ with $c(s) > \lambda_1$, providing a critical point different from $-\varphi_1$ and φ_1.

(ii) Thanks to Lemma 2.1 and part (i), we infer that $(s + c(s), c(s)) \in \widehat{\Sigma}_p$. □

3. The first nontrivial curve

The results in Section 2 permit us to determine the beginning of the spectrum $\widehat{\Sigma}_p$. We start by establishing that the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ are isolated in $\widehat{\Sigma}_p$. This is known from [15, Proposition 3.4] for Dirichlet problems and from [24, Proposition 3.1] for Steklov problems.

Proposition 3.1. *There exists no sequence $(a_n, b_n) \in \widehat{\Sigma}_p$ with $a_n > \lambda_1$ and $b_n > \lambda_1$ such that $(a_n, b_n) \to (a, b)$ with $a = \lambda_1$ or $b = \lambda_1$.*

Proof. Proceeding indirectly, assume that there exist sequences $(a_n, b_n) \in \widehat{\Sigma}_p$ and $(u_n) \subset W^{1,p}(\Omega)$ with the properties: $a_n \to \lambda_1$, $b_n \to b$, $a_n > \lambda_1$, $b_n > \lambda_1$, $\|u_n\|_{L^p(\Omega)} = 1$ and

$$\begin{aligned} -\Delta_p u_n &= a_n(u_n^+)^{p-1} - b_n(u_n^-)^{p-1} \quad \text{in } \Omega, \\ |\nabla u_n|^{p-2}\frac{\partial u_n}{\partial \nu} &= -\beta|u_n|^{p-2}u_n \qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

If we test (3.1) with $v = u_n$ (see (1.2)), we get

$$\|\nabla u_n\|^p_{L^p(\Omega)} = a_n \int_\Omega (u_n^+)^p \mathrm{d}x + b_n \int_\Omega (u_n^-)^p \mathrm{d}x - \beta \int_{\partial\Omega} |u_n|^p \mathrm{d}\sigma \le a_n + b_n,$$

which proves the boundedness of (u_n) in $W^{1,p}(\Omega)$. Hence, along a subsequence, $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$ and $L^p(\partial\Omega)$. Now, testing (3.1) with $\varphi = u_n - u$, we infer that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u)\mathrm{d}x = 0.$$

The $(S)_+$-property of $-\Delta_p$ on $W^{1,p}(\Omega)$ yields that $u_n \to u$ in $W^{1,p}(\Omega)$. Thus, u is a solution of the equation

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \mathrm{d}x = \lambda_1 \int_\Omega (u^+)^{p-1} v \mathrm{d}x - b \int_\Omega (u^-)^{p-1} v \mathrm{d}x - \beta \int_{\partial\Omega} |u|^{p-2} u v \mathrm{d}\sigma, \quad \forall v \in W^{1,p}(\Omega). \tag{3.2}$$

Inserting $v = u^+$ in (3.2) leads to

$$\int_\Omega |\nabla u^+|^p \mathrm{d}x = \lambda_1 \int_\Omega (u^+)^p \mathrm{d}x - \beta \int_{\partial\Omega} (u^+)^p \mathrm{d}\sigma.$$

This, in conjunction with the characterization of λ_1 in Section 1 and since $\|u\|_{L^p(\Omega)} = 1$, ensures that either $u^+ = 0$ or $u^+ = \varphi_1$. If $u^+ = 0$, then $u \le 0$ and (3.2) implies that u is an eigenfunction. Recalling that λ_1 is the only eigenfunction that does not change sign, we deduce that $u = -\varphi_1$ (see [1] and also Proposition 4.1). Consequently, this renders that (u_n) converges either to φ_1 or to $-\varphi_1$ in $L^p(\Omega)$, which forces us to have

$$\text{either } |\{x \in \Omega : u_n(x) < 0\}| \to 0 \text{ or } |\{x \in \Omega : u_n > 0\}| \to 0, \tag{3.3}$$

respectively, where $|\cdot|$ denotes the Lebesgue measure. Indeed, assuming for instance $u_n \to \varphi_1$ in $L^p(\Omega)$, since for any compact subset $K \subset \Omega$ there holds

$$\int_{\{u_n<0\}\cap K} |u_n - \varphi_1|^p \mathrm{d}x \ge \int_{\{u_n<0\}\cap K} \varphi_1^p \mathrm{d}x \ge C|\{u_n < 0\} \cap K|,$$

with a constant $C > 0$, it is seen that the first assertion in (3.3) is fulfilled.

On the other hand, using $v = u_n^+$ as test function for (3.1) in conjunction with the Hölder inequality and the continuity of the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, with $p < q \le p^*$, we obtain the estimate

$$\begin{aligned}\int_\Omega |\nabla u_n^+|^p \mathrm{d}x + \int_\Omega (u_n^+)^p \mathrm{d}x &= a_n \int_\Omega (u_n^+)^p \mathrm{d}x - \beta \int_{\partial\Omega} (u_n^+)^p \mathrm{d}\sigma + \int_\Omega (u_n^+)^p \mathrm{d}x \\ &\le (a_n + 1) \int_\Omega (u_n^+)^p \mathrm{d}x \\ &\le (a_n + 1)C|\{x \in \Omega : u_n(x) > 0\}|^{1-\frac{p}{q}} \|u_n^+\|^p_{W^{1,p}(\Omega)},\end{aligned}$$

with a constant $C > 0$. We infer that

$$|\{x \in \Omega : u_n(x) > 0\}|^{1-\frac{p}{q}} \ge (a_n + 1)^{-1} C^{-1}$$

and in the same way,

$$|\{x \in \Omega : u_n(x) < 0\}|^{1-\frac{p}{q}} \ge (b_n + 1)^{-1} C^{-1}.$$

Since $(a_n, b_n) \in \widehat{\Sigma}_p$ does not belong to the trivial lines of $\widehat{\Sigma}_p$, we have that u_n changes sign. Hence, through the inequalities above, we reach a contradiction with (3.3), which completes the proof. □

The following auxiliary fact is helpful to link with the results established in Section 2.

Lemma 3.2. *For every $r > \inf_S J_s = \lambda_1 - s$, each connected component of $\{u \in S : J_s(u) < r\}$ contains a critical point, in fact a local minimizer of $\widetilde{J}_s$.*

Proof. Let C be a connected component of $\{u \in S : J_s(u) < r\}$ and denote $d = \inf\{J_s(u) : u \in \overline{C}\}$. We claim that there exists $u_0 \in \overline{C}$ such that $\widetilde{J}_s(u_0) = d$. To this end, let $(u_n) \subset C$ be a sequence such that $\widetilde{J}_s(u_n) \le d + \frac{1}{n^2}$. Applying Ekeland's variational principle to $\widetilde{J}_s$ on $\overline{C}$ provides a sequence $(v_n) \subset \overline{C}$ such that

$$\widetilde{J}_s(v_n) \le \widetilde{J}_s(u_n), \tag{3.4}$$

$$\|u_n - v_n\|_{W^{1,p}(\Omega)} \le \frac{1}{n}, \tag{3.5}$$

$$\widetilde{J}_s(v_n) \leq \widetilde{J}_s(v) + \frac{1}{n}\|v - v_n\|_{W^{1,p}(\Omega)}, \quad \forall v \in \overline{C}. \tag{3.6}$$

If n is sufficiently large, by (3.4) we obtain

$$\widetilde{J}_s(v_n) \leq \widetilde{J}_s(u_n) \leq d + \frac{1}{n^2} < r.$$

Moreover, owing to (3.6), it can be shown that (v_n) is a Palais–Smale sequence for $\widetilde{J}_s$. Then Lemma 2.5 and (3.5) ensure that, up to a relabeled subsequence, $u_n \to u_0$ in $W^{1,p}(\Omega)$ with $u_0 \in \overline{C}$ and $\widetilde{J}_s(v) = d$.

We note that $u_0 \notin \partial C$ because otherwise the maximality of C as a connected component would be contradicted, so u_0 is a local minimizer of J_s and we are done. □

Recall from Proposition 2.7 that it was constructed a curve $(s + c(s), c(s)) \in \widehat{\Sigma}_p$ for $s \geq 0$. As $\widehat{\Sigma}_p$ is symmetric with respect to the diagonal, we can complete it with its symmetric part obtaining the following curve in $\widehat{\Sigma}_p$:

$$\mathcal{C} := \{(s + c(s), c(s)), (c(s), s + c(s)) : s \geq 0\}. \tag{3.7}$$

The next result points out that $\mathcal{C}$ is the first nontrivial curve in $\widehat{\Sigma}_p$.

Theorem 3.3. *Let $s \geq 0$. Then $(s+c(s), c(s)) \in \mathcal{C}$ is the first point in the intersection between $\widehat{\Sigma}_p$ and the ray $(s, 0)+t(1, 1)$, $t > \lambda_1$.*

Proof. Assume, by contradiction, the existence of a point $(s + \mu, \mu) \in \widehat{\Sigma}_p$ with $\lambda_1 < \mu < c(s)$. Proposition 3.1 and the fact that $\widehat{\Sigma}_p$ is closed enable us to suppose that μ is the minimum number with the required property. By virtue of Lemma 2.1, μ is a critical value of the functional $\widetilde{J}_s$ and there is no critical value of $\widetilde{J}_s$ in the interval (λ_1, μ). We complete the proof by reaching a contradiction to the definition of $c(s)$ in (2.4). To this end, it suffices to construct a path in Γ along which there holds $\widetilde{J}_s \leq \mu$.

Let $u \in S$ be a critical point of $\widetilde{J}_s$ with $\widetilde{J}_s(u) = \mu$. Then u fulfills

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \mathrm{d}x = (s + \mu)\int_\Omega (u^+)^{p-1} v \mathrm{d}x - \mu \int_\Omega (u^-)^{p-1} v \mathrm{d}x - \beta \int_{\partial\Omega} |u|^{p-2} uv \mathrm{d}\sigma, \quad \forall v \in W^{1,p}(\Omega).$$

Setting $v = u^+$ and $v = -u^-$ yields

$$\int_\Omega |\nabla u^+|^p \mathrm{d}x = (s + \mu)\int_\Omega (u^+)^p \mathrm{d}x - \beta \int_{\partial\Omega} (u^+)^p \mathrm{d}\sigma \tag{3.8}$$

and

$$\int_\Omega |\nabla u^-|^p \mathrm{d}x = \mu \int_\Omega (u^-)^p \mathrm{d}x - \beta \int_{\partial\Omega} (u^-)^p \mathrm{d}\sigma, \tag{3.9}$$

respectively. Since u changes sign (see Proposition 4.1), the following paths are well defined on S:

$$u_1(t) = \frac{(1-t)u + tu^+}{\|(1-t)u + tu^+\|_{L^p(\Omega)}}, \qquad u_2(t) = \frac{(1-t)u^+ + tu^-}{\|(1-t)u^+ + tu^-\|_{L^p(\Omega)}}, \qquad u_3(t) = \frac{-tu^- + (1-t)u}{\|-tu^- + (1-t)u\|_{L^p(\Omega)}},$$

for all $t \in [0, 1]$. By means of direct calculations based on (3.8) and (3.9) we infer that

$$\widetilde{J}_s(u_1(t)) = \widetilde{J}_s(u_3(t)) = \mu, \quad \text{for all } t \in [0, 1]$$

and

$$\widetilde{J}_s(u_2(t)) = \mu - \frac{st^p\|u^-\|^p_{L^p(\Omega)}}{\|(1-t)u^+ + tu^-\|^p_{L^p(\Omega)}} \leq \mu, \quad \text{for all } t \in [0, 1].$$

Due to the minimality property of μ, the only critical points of $\widetilde{J}_s$ in the set $\{w \in S : \widetilde{J}_s(w) < \mu - s\}$ are φ_1 and possibly $-\varphi_1$ provided $\mu - s > \lambda_1$. We note that, because $u^-/\|u^-\|_{L^p(\Omega)}$ does not change sign and vanishes on a set of positive measure, it is not a critical point of $\widetilde{J}_s$. Therefore, there exists a C^1 path $\alpha : [-\varepsilon, \varepsilon] \to S$ with $\alpha(0) = u^-/\|u^-\|_{L^p(\Omega)}$ and $\mathrm{d}/\mathrm{d}t\widetilde{J}_s(\alpha(t))|_{t=0} \neq 0$. Using this path and observing from (3.9) that $\widetilde{J}_s(u^-/\|u^-\|_{L^p(\Omega)}) = \mu - s$, we can move from $u^-/\|u^-\|_{L^p(\Omega)}$ to a point v with $\widetilde{J}_s(v) < \mu - s$. Applying Lemma 3.2 (see also [15, Lemma 3.5]), we find that the connected component of $\{w \in S : \widetilde{J}_s(w) < \mu - s\}$ containing v crosses $\{\varphi_1, -\varphi_1\}$. Let us say that it passes through φ_1, otherwise the reasoning is the same employing $-\varphi_1$. Consequently, there is a path $u_4(t)$ from $u^-/\|u^-\|_{L^p(\Omega)}$ to φ_1 within the set $\{w \in S : \widetilde{J}_s(w) \leq \mu - s\}$. Then the path $-u_4(t)$ joins $-u^-/\|u^-\|_{L^p(\Omega)}$ and $-\varphi_1$ and, since $u_4(t) \in S$, we have

$$\widetilde{J}_s(-u_4(t)) \leq \widetilde{J}_s(u_4(t)) + s \leq \mu - s + s = \mu \quad \text{for all } t.$$

Connecting $u_1(t)$, $u_2(t)$ and $u_4(t)$, we construct a path joining u and φ_1, and joining $u_3(t)$ and $-u_4(t)$ we get a path which connects u and $-\varphi_1$. These yield a path $\gamma(t)$ on S joining φ_1 and $-\varphi_1$. Furthermore, in view of the discussion above, it turns out that $\widetilde{J}_s(\gamma(t)) \leq \mu$ for all t. This proves the theorem. □

Corollary 3.4. *The second eigenvalue λ_2 of (1.3) has the variational characterization given in (1.4).*

Proof. Theorem 3.3 for $s = 0$ ensures that $c(0) = \lambda_2$. The conclusion now follows by applying Proposition 2.7 (i) with $s = 0$. □

4. Properties of the first curve

The following proposition establishes an important sign property related to the curve $\mathcal{C}$ in (3.7).

Proposition 4.1. *Let $(a_0, b_0) \in \mathcal{C}$ and $a, b \in L^\infty(\Omega)$ satisfy $\lambda_1 \leq a(x) \leq a_0$, $\lambda_1 \leq b(x) \leq b_0$ for a.a. $x \in \Omega$ such that $\lambda_1 < a(x)$ and $\lambda_1 < b(x)$ on subsets of positive measure. Then any nontrivial solution u of*

$$\begin{aligned} -\Delta_p u &= a(x)(u^+)^{p-1} - b(x)(u^-)^{p-1} && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= -\beta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

changes sign in Ω.

Proof. Let u be a nontrivial solution of Eq. (4.1). Then, $-u$ is a nontrivial solution of

$$\begin{aligned} -\Delta_p z &= b(x)(z^+)^{p-1} - a(x)(z^-)^{p-1} && \text{in } \Omega, \\ |\nabla z|^{p-2}\frac{\partial z}{\partial \nu} &= -\beta|z|^{p-2}z && \text{on } \partial\Omega, \end{aligned}$$

hence, we can suppose that the point $(a_0, b_0) \in \mathcal{C}$ is such that $a_0 \geq b_0$.

We argue by contradiction and assume that u does not change sign in Ω. Without loss of generality, we may admit that $u \geq 0$ a.e. in Ω, so u is a solution of the Robin weighted eigenvalue problem with weight $a(x)$:

$$\begin{aligned} -\Delta_p u &= a(x)u^{p-1} && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= -\beta u^{p-1} && \text{on } \partial\Omega. \end{aligned}$$

It means that u is an eigenfunction corresponding to the eigenvalue 1 for this weighted problem. Recall that the first eigenvalue $\lambda_1(a)$ of the above weighted problem is expressed as

$$\lambda_1(a) = \inf_{\substack{v \in W^{1,p}(\Omega) \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^p \mathrm{d}x + \beta \int_{\partial\Omega} |v|^p \mathrm{d}\sigma}{\int_\Omega a(x)|v|^p \mathrm{d}x}.$$

The fact that $u \geq 0$ entails $\lambda_1(a) = 1$ because the only eigenvalue whose eigenfunction does not change sign is $\lambda_1(a)$ (see [1]). Then the hypothesis that $\lambda_1 < a(x)$ on a set of positive measure leads to the contradiction

$$1 = \frac{\int_\Omega |\nabla \varphi_1|^p \mathrm{d}x + \beta \int_{\partial\Omega} |\varphi_1|^p \mathrm{d}\sigma}{\lambda_1} > \frac{\int_\Omega |\nabla \varphi_1|^p \mathrm{d}x + \beta \int_{\partial\Omega} |\varphi_1|^p \mathrm{d}\sigma}{\int_\Omega a(x)\varphi_1^p \mathrm{d}x} \geq \lambda_1(a) = 1,$$

which completes the proof. □

Proposition 4.2. *The curve $s \mapsto (s + c(s), c(s))$ is Lipschitz continuous and decreasing.*

Proof. If $s_1 < s_2$, then it follows that $\widetilde{J}_{s_1}(u) \geq \widetilde{J}_{s_2}(u)$ for all $u \in S$, which ensures that $c(s_1) \geq c(s_2)$. For every $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that

$$\max_{u \in \gamma[-1,1]} \widetilde{J}_{s_2}(u) \leq c(s_2) + \varepsilon,$$

hence

$$0 \leq c(s_1) - c(s_2) \leq \max_{u \in \gamma[-1,1]} \widetilde{J}_{s_1}(u) - \max_{u \in \gamma[-1,1]} \widetilde{J}_{s_2}(u) + \varepsilon.$$

Taking $u_0 \in \gamma[-1, 1]$ such that

$$\max_{u \in \gamma[-1,1]} \widetilde{J}_{s_1}(u) = \widetilde{J}_{s_1}(u_0)$$

yields

$$0 \le c(s_1) - c(s_2) \le \widetilde{J}_{s_1}(u_0) - \widetilde{J}_{s_2}(u_0) + \varepsilon = s_1 - s_2 + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this ensures that $s \mapsto (s + c(s), c(s))$ is Lipschitz continuous.

In order to prove that the curve is decreasing, it suffices to argue for $s > 0$. Let $0 < s_1 < s_2$. Then, since $(s_1 + c(s_1), c(s_1))$, $(s_2 + c(s_2), c(s_2)) \in \widehat{\Sigma}_p$, Theorem 3.3 implies that $s_1 + c(s_1) < s_2 + c(s_2)$. On the other hand, as already remarked, there holds $c(s_1) \ge c(s_2)$, which completes the proof. □

Next we investigate the asymptotic behavior of the curve $\mathcal{C}$.

Theorem 4.3. *Let $p \le N$. Then the limit of $c(s)$ as $s \to +\infty$ is λ_1.*

Proof. Let us proceed by contradiction and suppose that $c(s)$ does not converge to λ_1 as $s \to +\infty$. Then there exists $\delta > 0$ such that

$$\max_{u \in \gamma[-1,1]} \widetilde{J}_s(u) \ge \lambda_1 + \delta \quad \text{for all } \gamma \in \Gamma \text{ and all } s \ge 0.$$

Since $p \le N$, we can choose a function $\psi \in W^{1,p}(\Omega)$ which is unbounded from above. Then we define $\gamma \in \Gamma$ by

$$\gamma(t) = \frac{t\varphi_1 + (1 - |t|)\psi}{\|t\varphi_1 + (1 - |t|)\psi\|_{L^p(\Omega)}}, \quad t \in [-1, 1].$$

For every $s > 0$, let $t_s \in [-1, 1]$ satisfy

$$\max_{t \in [-1,1]} \widetilde{J}_s(\gamma(t)) = \widetilde{J}_s(\gamma(t_s)).$$

Denoting $v_s = t_s\varphi_1 + (1 - |t_s|)\psi$, we infer that

$$\int_\Omega |\nabla v_s|^p dx + \beta \int_{\partial\Omega} |v_s|^p d\sigma - s \int_\Omega (v_s^+)^p dx \ge (\lambda_1 + \delta) \int_\Omega |v_s|^p dx. \tag{4.2}$$

Letting $s \to +\infty$, we can assume along a subsequence that $t_s \to \widetilde{t} \in [-1, 1]$. The family v_s being bounded in $W^{1,p}(\Omega)$, from (4.2) one sees that

$$\int_\Omega (v_s^+)^p dx \to 0 \quad \text{as } s \to +\infty,$$

which forces

$$\widetilde{t}\varphi_1 + (1 - |\widetilde{t}|)\psi \le 0.$$

Due to the choice of ψ, this is impossible unless $\widetilde{t} = -1$. Passing to the limit in (4.2) as $s \to +\infty$ and using $\widetilde{t} = -1$, we arrive at the contradiction $\delta \le 0$, so the proof is complete. □

It remains to study the asymptotic properties of the curve $\mathcal{C}$ when $p > N$. For $\beta = 0$, problem (1.1) becomes a Neumann problem with homogeneous boundary condition that was studied in [22]. Therein, it is shown that

$$\lim_{s \to +\infty} c(s) = \begin{cases} \lambda_1 = 0 & \text{if } p \le N \\ \widetilde{\lambda} & \text{if } p > N, \end{cases}$$

where

$$\widetilde{\lambda} = \inf \left\{ \int_\Omega |\nabla u|^p dx : u \in W^{1,p}(\Omega),\ \|u\|_{L^p(\Omega)} = 1 \text{ and } u \text{ vanishes somewhere in } \overline{\Omega} \right\}.$$

Therefore, we only have to treat the case $\beta > 0$. In this respect, the key idea is to work with an adequate equivalent norm on the space $W^{1,p}(\Omega)$. So, for $\beta > 0$ we introduce the norm

$$\|u\|_\beta = \|\nabla u\|_{L^p(\Omega)} + \beta \|u\|_{L^p(\partial\Omega)}, \tag{4.3}$$

which is an equivalent norm on $W^{1,p}(\Omega)$ (see also [28, Theorem 2.1]). Then we have the following.

Theorem 4.4. *Let $\beta > 0$ and $p > N$. Then the limit of $c(s)$ as $s \to +\infty$ is*

$$\overline{\lambda} = \inf_{u \in L} \max_{r \in \mathbb{R}} \frac{\int_\Omega |\nabla(r\varphi_1 + u)|^p dx + \beta \int_{\partial\Omega} |r\varphi_1 + u|^p d\sigma}{\int_\Omega |r\varphi_1 + u|^p dx},$$

where

$$L = \{u \in W^{1,p}(\Omega) : u \text{ vanishes somewhere in } \overline{\Omega},\ u \not\equiv 0\}.$$

Moreover, there holds $\overline{\lambda} > \lambda_1$.

Proof. First, we are going to prove the strict inequality $\overline{\lambda} > \lambda_1$. Since for every $w \in L$ one has

$$\frac{\int_\Omega |\nabla w|^p \mathrm{d}x + \beta \int_{\partial\Omega} |w|^p \mathrm{d}\sigma}{\int_\Omega |w|^p \mathrm{d}x} \leq \max_{r\in\mathbb{R}} \frac{\int_\Omega |\nabla(r\varphi_1 + w)|^p \mathrm{d}x + \beta \int_{\partial\Omega} |r\varphi_1 + w|^p \mathrm{d}\sigma}{\int_\Omega |r\varphi_1 + w|^p \mathrm{d}x},$$

we conclude that

$$\lambda_1 \leq \inf_{w\in L} \frac{\int_\Omega |\nabla w|^p \mathrm{d}x + \beta \int_{\partial\Omega} |w|^p \mathrm{d}\sigma}{\int_\Omega |w|^p \mathrm{d}x} \leq \overline{\lambda}. \tag{4.4}$$

Let us check that the first inequality in (4.4) is strict. On the contrary, we would find a sequence $(w_n) \subset L$ satisfying

$$\frac{\int_\Omega |\nabla w_n|^p \mathrm{d}x + \beta \int_{\partial\Omega} |w_n|^p \mathrm{d}\sigma}{\int_\Omega |w_n|^p \mathrm{d}x} \to \lambda_1 \quad \text{as } n \to \infty.$$

Set $v_n = \frac{w_n}{\|w_n\|_\beta}$, where $\|\cdot\|_\beta$ denotes the equivalent norm on $W^{1,p}(\Omega)$ introduced in (4.3). We note that $\|v_n\|_\beta = 1$ and

$$\frac{1}{\int_\Omega |v_n|^p \mathrm{d}x} \to \lambda_1 \quad \text{as } n \to \infty.$$

Due to the compact embedding $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$, there is a subsequence of (v_n), still denoted by (v_n), such that $v_n \rightharpoonup v$ in $W^{1,p}(\Omega)$ and $v_n \to v$ uniformly on $\overline{\Omega}$. It follows that $v \in L$ and

$$\frac{\int_\Omega |\nabla v|^p \mathrm{d}x + \beta \int_{\partial\Omega} |v|^p \mathrm{d}\sigma}{\int_\Omega |v|^p \mathrm{d}x} \leq \lambda_1 = \frac{1}{\int_\Omega |v|^p \mathrm{d}x},$$

which ensures that v is an eigenfunction in (1.3) corresponding to the first eigenvalue λ_1. This is a contradiction because every eigenfunction associated to λ_1 is strictly positive or negative on $\overline{\Omega}$, whereas $v \in L$. Hence, recalling (4.4), we get $\overline{\lambda} > \lambda_1$.

Now we prove the first part in the theorem. We start by claiming that there exist $u \in L$ such that

$$\max_{r\in\mathbb{R}} \frac{\int_\Omega |\nabla(r\varphi_1 + u)|^p \mathrm{d}x + \beta \int_{\partial\Omega} |r\varphi_1 + u|^p \mathrm{d}\sigma}{\int_\Omega |r\varphi_1 + u|^p \mathrm{d}x} = \overline{\lambda}. \tag{4.5}$$

By the definition of $\overline{\lambda}$, we can find sequences $(u_n) \subset L$ and $(r_n) \subset \mathbb{R}$ such that

$$\begin{aligned} &\max_{r\in\mathbb{R}} \frac{\int_\Omega |\nabla(r\varphi_1 + u_n)|^p \mathrm{d}x + \beta \int_{\partial\Omega} |r\varphi_1 + u_n|^p \mathrm{d}\sigma}{\int_\Omega |r\varphi_1 + u_n|^p \mathrm{d}x} \\ &= \frac{\int_\Omega |\nabla(r_n\varphi_1 + u_n)|^p \mathrm{d}x + \beta \int_{\partial\Omega} |r_n\varphi_1 + u_n|^p \mathrm{d}\sigma}{\int_\Omega |r_n\varphi_1 + u_n|^p \mathrm{d}x} \to \overline{\lambda} \quad \text{as } n \to \infty. \end{aligned} \tag{4.6}$$

Without loss of generality, we can assume that $\|u_n\|_{W^{1,p}(\Omega)} = 1$. The sequence (r_n) has to be bounded because otherwise there would exist a relabeled subsequence $r_n \to +\infty$, which results in

$$\frac{\int_\Omega |\nabla r_n\varphi_1 + u_n|^p \mathrm{d}x + \beta \int_{\partial\Omega} |r_n\varphi_1 + u_n|^p \mathrm{d}\sigma}{\int_\Omega |r_n\varphi_1 + u_n|^p \mathrm{d}x} \to \lambda_1.$$

This implies that $\lambda_1 = \overline{\lambda}$, contradicting the inequality $\overline{\lambda} > \lambda_1$. Therefore, we may suppose that $r_n \to \widetilde{r} \in \mathbb{R}$ and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ as well as $u_n \to u$ uniformly in $\overline{\Omega}$, with some $u \in L$. Then, through (4.6) and the definition of $\overline{\lambda}$, we see that (4.5) holds true.

To prove that $c(s) \to \overline{\lambda}$ as $s \to +\infty$, we argue by contradiction admitting that there exists $\delta > 0$ such that

$$\max_{t\in[-1,1]} \widetilde{J}_s(\gamma(t)) \geq \overline{\lambda} + \delta \quad \text{for all } \gamma \in \Gamma \text{ and all } s \geq 0.$$

Here the decreasing monotonicity of $c(s)$ has been used (see Proposition 4.2). Consider the path $\gamma \in \Gamma$ defined by

$$\gamma(t) = \frac{t\varphi_1 + (1-|t|)u}{\|t\varphi_1 + (1-|t|)u\|_{L^p(\Omega)}}, \quad t \in [-1, 1],$$

with u given in (4.5). Proceeding as in the proof of Theorem 4.3, for every $s > 0$ we fix $t_s \in [-1, 1]$ to satisfy

$$\max_{t\in[-1,1]} \widetilde{J}_s(\gamma(t)) = \widetilde{J}_s(\gamma(t_s))$$

and denote $v_s = t_s\varphi_1 + (1 - |t_s|)u$. We have

$$\int_\Omega |\nabla v_s|^p \mathrm{d}x + \beta \int_{\partial\Omega} |v_s|^p \mathrm{d}\sigma - s \int_\Omega (v_s^+)^p \mathrm{d}x \geq (\overline{\lambda} + \delta) \int_\Omega |v_s|^p \mathrm{d}x. \tag{4.7}$$

From (4.7) and since v_s is uniformly bounded, we obtain $\int_\Omega (v_s^+)^p \mathrm{d}x \to 0$ and $t_s \to \widetilde{t} \in [-1, 1]$ as $s \to +\infty$, which yields $\widetilde{t}\varphi_1 \leq -(1 - |\widetilde{t}|)u$. As $\varphi_1 > 0$ and u vanishes somewhere in $\overline{\Omega}$, we deduce that $\widetilde{t} \leq 0$. In addition, passing to the limit in (4.7) leads to

$$\int_\Omega |\nabla(\widetilde{t}\varphi_1 + (1 - |\widetilde{t}|)u)|^p \mathrm{d}x + \beta \int_{\partial\Omega} |\widetilde{t}\varphi_1 + (1 - |\widetilde{t}|)u|^p \mathrm{d}\sigma \geq (\overline{\lambda} + \delta) \int_\Omega |\widetilde{t}\varphi_1 + (1 - |\widetilde{t}|)u|^p \mathrm{d}x. \tag{4.8}$$

If $\widetilde{t} \neq -1$, (4.8) can be expressed as

$$\frac{\int_\Omega \left|\nabla\left(\frac{\widetilde{t}}{1+\widetilde{t}}\varphi_1 + u\right)\right|^p \mathrm{d}x + \beta \int_{\partial\Omega} \left|\frac{\widetilde{t}}{1+\widetilde{t}}\varphi_1 + u\right|^p \mathrm{d}\sigma}{\int_\Omega \left|\frac{\widetilde{t}}{1+\widetilde{t}}\varphi_1 + u\right|^p \mathrm{d}x} \geq \overline{\lambda} + \delta.$$

Comparing with (4.5) reveals that a contradiction is reached. If $\widetilde{t} = -1$, in view of (4.8) and $\overline{\lambda} > \lambda_1$, we also arrive at a contradiction, which establishes the result. □

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MULTIPLICITY RESULTS FOR A CLASS OF ELLIPTIC PROBLEMS WITH NONLINEAR BOUNDARY CONDITION

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Abstract. This paper provides multiplicity results for a class of nonlinear elliptic problems under a nonhomogeneous Neumann boundary condition. We prove the existence of three nontrivial solutions to these problems which depend on the Fučík spectrum of the negative p-Laplacian with a Robin boundary condition. Using variational and topological arguments combined with an equivalent norm on the Sobolev space $W^{1,p}$ it is obtained a smallest positive solution, a greatest negative solution, and a sign-changing solution.

1. **Introduction.** The purpose of this article is to investigate the existence and multiplicity of weak solutions to elliptic equations with nonhomogeneous Neumann boundary condition. Specifically, given a bounded domain $\Omega \subseteq \mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and let $1 < p < \infty$, we consider the problem

$$\begin{aligned} -\Delta_p u &= a(u^+)^{p-1} - b(u^-)^{p-1} + f(x,u) && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= h(x,u) - \theta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the negative p-Laplacian, $\partial u/\partial\nu$ denotes the outer normal derivative of u while the values a, b and θ are real parameters specified later. The terms $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$ stand for the positive and negative part of u, respectively, and the perturbations, namely $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, are some Carathéodory functions satisfying suitable hypotheses, see (H) below. For the sake of simplicity we omit the denotation for the trace operator $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ which is applied to the functions on the boundary $\partial\Omega$.

The main goal of this article is to prove the existence of three nontrivial weak solutions of the nonhomogeneous Neumann boundary value problem given in (1.1). More precisely, we establish two extremal constant-sign solutions, namely a smallest positive solution u_+ as well as a greatest negative solution u_-, and finally, the existence of a nontrivial sign-changing solution u_0 lying between these extremal constant-sign solutions is pointed out.

Throughout the paper we impose the following assumptions.

(H) Let $\theta > 0$ be a fixed constant and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions satisfying the subsequent conditions:

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(H1) f is bounded on bounded sets.
(H2)

$$\lim_{|s|\to\infty} \frac{f(x,s)}{|s|^{p-2}s} = -\infty, \quad \text{uniformly with respect to a.a. } x \in \Omega.$$

(H3)

$$\lim_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} = 0, \quad \text{uniformly with respect to a.a. } x \in \Omega.$$

(H4) h is bounded on bounded sets.
(H5) There exists a number $s_\theta > 0$ such that

$$\frac{h(x,s)}{|s|^{p-2}s} < \theta, \quad \text{for a.a. } x \in \partial\Omega \text{ and for all } |s| > s_\theta.$$

(H6)

$$\lim_{s\to 0} \frac{h(x,s)}{|s|^{p-2}s} = 0, \quad \text{uniformly with respect to a.a. } x \in \partial\Omega.$$

(H7) h satisfies the condition

$$|h(x_1,s_1) - h(x_2,s_2)| \le L\Big[|x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha\Big],$$

for all pairs $(x_1,s_1),(x_2,s_2)$ in $\partial\Omega \times [-K,K]$, where K is a positive constant and $\alpha \in (0,1]$.

By means of the hypotheses (H3) and (H6) we see at once that $f(x,0) = h(x,0) = 0$ reasoning that $u \equiv 0$ is a trivial solution of (1.1). The condition (H7) is a Hölder continuity assumption which is needed to make use of the $C^{1,\alpha}$-regularity of Lieberman (see [20]).

In a recent work of the author [32] there are shown multiplicity results to equations of the form

$$\begin{aligned} -\Delta_p u &= f(x,u) - |u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= a(u^+)^{p-1} - b(u^-)^{p-1} + g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where the solutions of (1.2) depend on the so-called Steklov Fučík spectrum of the negative p-Laplacian which was intensively treated by Martínez and Rossi in [22]. The novelty of this paper is on the one hand that the solutions of (1.1) depend on the Robin Fučík spectrum of $-\Delta_p$ (see Section 2 for a detailed introduction) and on the other hand we could drop a hypothesis on the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, which was required in [32], namely

(A1) There exists a number $\delta_f > 0$ such that $\dfrac{f(x,s)}{|s|^{p-2}s} \ge 0$ for all $0 < |s| \le \delta_f$ and for a.a. $x \in \Omega$.

Assumption (A1) means that the function f must change sign near zero. Now, we do not need this condition on f. Further, regarding the behaviour at infinity, the boundary function in [32] has to satisfy the condition

(A2) $\lim_{|s|\to\infty} \frac{g(x,s)}{|s|^{p-2}s} = -\infty$ uniformly with respect to a.a. $x \in \partial\Omega$.

We point out that we can replace (A2) by the weaker condition (H5).
Another novelty is the usage of an equivalent norm on the space $W^{1,p}(\Omega)$ obtained by Deng (see [12]) which contains the norm $\|\cdot\|_{L^p(\partial\Omega)}$ instead of $\|\cdot\|_{L^p(\Omega)}$. This

ensures, in particular, that suitable energy functionals concerning problem (1.1) (involving appropriate truncation functions to make sure the finiteness of the integrals) satisfy the coercivity and the Palais-Smale condition which is required in our approach. It should be mentioned that we do not need differentiability, polynomial growth, or some integral conditions on the mappings f and h. In order to prove our main results we make use of variational and topological tools, e.g. critical point theory, the mountain-pass theorem, the second deformation lemma and the so-called Robin Fučík spectrum of the negative p-Laplacian.

Elliptic equations with a nonhomogeneous Neumann boundary condition regarding existence and multiplicity of solutions were studied by a number of authors in the last years. Without guarantee of completeness we refer to the papers in [1], [13], [14], [15], [19], [21], [23], [29], [35], and the references therein. With reference to homogeneous Neumann problems, multiple solution results can be found for example in [2], [4], [5] and [6]. In the Dirichlet case there also exists a number of publications according to the subject of multiplicity results, see e.g. in [9], [10], [11], and [17].

The paper is organized as follows. In Section 2 we give the basic notations including the definition of a sub- and supersolution of (1.1), we point out some recent results with regard to the Robin Fučík spectrum of the negative p-Laplacian and we consider a second auxiliary problem which is needed in our treatment. The third section is devoted to the proofs of the existence of specific sub- and supersolutions of (1.1) which leads to the existence of two ordered pairs of sub- and supersolution, one with positive sign and the other one with negative sign. Then, we can derive the existence of two constant-sign solutions thanks to the method of sub- and supersolution dealt in [7]. The existence of extremal constant-sign solutions, more exact a smallest positive solution and a greatest negative solution of (1.1), is shown in Section 4 using functional analytical arguments in association with the properties of the Robin Fučík spectrum of $-\Delta_p$. In the last section we prove the existence of a sign-changing solution applying the fact that every nontrivial solution between the obtained extremal constant-sign solution must be a sign-changing solution provided it is unequal to these extremal solutions. Variational and topological tools like the mountain-pass theorem, critical point theory and the second deformation lemma are found a use in this last section.

2. **Preliminaries.** By $L^p(\Omega), L^p(\partial\Omega)$ and $W^{1,p}(\Omega)$ we denote the usual Lebesgue and Sobolev spaces with their norms $\|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{L^p(\partial\Omega)}$ and $\|\cdot\|_{W^{1,p}(\Omega)}$, respectively. Given $\zeta > 0$ we introduce an equivalent norm on $W^{1,p}(\Omega)$ given by

$$\|u\|_\zeta = \|\nabla u\|_{L^p(\Omega)} + \zeta\|u\|_{L^p(\partial\Omega)} \tag{2.1}$$

(see e.g. Deng [12]). We say that $u \in W^{1,p}(\Omega)$ is a weak solution of problem (1.1) if

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx\\ &= \int_\Omega (a(u^+)^{p-1} - b(u^-)^{p-1} + f(x,u))v dx + \int_{\partial\Omega} (h(x,u) - \theta|u|^{p-2}u)v d\mu,\end{aligned}$$

holds for all test functions $v \in W^{1,p}(\Omega)$ while $d\mu$ denotes the usual $(N-1)$-dimensional surface measure. Further, the definition of weak sub- and supersolutions is required in our treatments. A function $\underline{u} \in W^{1,p}(\Omega)$ is said to be a weak

subsolution of problem (1.1) if the inequality

$$\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}\cdot\nabla v dx$$
$$\leq \int_\Omega (a(\underline{u}^+)^{p-1}-b(\underline{u}^-)^{p-1}+f(x,\underline{u}))v dx+\int_{\partial\Omega}(h(x,\underline{u})-\theta|\underline{u}|^{p-2}\underline{u})v d\mu,$$

is satisfied for all nonnegative test functions $v\in W^{1,p}(\Omega)$. Analogously, replacing '$\underline{u}$' by '$\overline{u}$' and '$\leq$' by '$\geq$', we obtain the definition of a weak supersolution of problem (1.1). It is obvious that every weak solution is both a weak subsolution and a weak supersolution. As a consequence of the assumptions in (H) we get a helpful result stated above.

Corollary 2.1. *Under the hypothesis (H) for each $\xi>0$ there exist constants $\chi_1,\chi_2>0$ such that, for all $0\leq|s|\leq\xi$,*

$$|f(x,s)|\leq\chi_1|s|^{p-1},\ \textit{for a.a. } x\in\Omega,\quad |h(x,s)|\leq\chi_2|s|^{p-1},\ \textit{for a.a. } x\in\partial\Omega.$$

To be more precise, the growth conditions in Corollary 2.1 come from the assumptions (H1), (H3), (H4) and (H6), respectively.

As mentioned in the Introduction, we need the properties of the Fučík spectrum of the negative p-Laplacian $-\Delta_p$ with Robin boundary condition. This spectrum is defined as the set $\widehat{\Sigma}_p$ of all pairs $(a,b)\in\mathbb{R}^2$ such that

$$\begin{aligned} -\Delta_p u &= a(u^+)^{p-1}-b(u^-)^{p-1} && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} &= -\beta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

is solved nontrivially meaning that $u\in W^{1,p}(\Omega), u\not\equiv 0$, and verifies the equality

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx+\beta\int_{\partial\Omega}|u|^{p-2}uv d\mu=\int_\Omega (a(u^+)^{p-1}-b(u^-)^{p-1})v dx, \tag{2.3}$$

for all $v\in W^{1,p}(\Omega)$. In (2.2), respectively (2.3), the parameter β is supposed to be a fixed, nonnegative constant. If $\beta=0$, (2.2) reduces to the Fučík spectrum Θ_p of the negative Neumann p-Laplacian (see [3]). The special case $a=b=\lambda$ leads to

$$-\Delta_p u=\lambda|u|^{p-2}u \quad\text{in } \Omega,\qquad |\nabla u|^{p-2}\frac{\partial u}{\partial\nu}=-\beta|u|^{p-2}u \quad\text{on } \partial\Omega, \tag{2.4}$$

which is known as the Robin eigenvalue problem of the negative p-Laplacian. Problem (2.4) was studied in the important publication of Lê [18] devoted to the eigenvalue problems for the negative p-Laplacian. In the Robin case he proved that the first eigenvalue λ_1 of (2.4) corresponding to the fixed value β is simple, isolated and it can be variationally characterized through

$$\lambda_1=\inf_{u\in W^{1,p}(\Omega)}\left\{\int_\Omega|\nabla u|^p dx+\beta\int_{\partial\Omega}|u|^p d\mu:\int_\Omega|u|^p dx=1\right\}. \tag{2.5}$$

Moreover, the set of eigenvalues to (2.4) is closed (see [18, Theorem 5.9]). It is also known that the first eigenfunction φ_1 associated to λ_1 has constant sign in Ω and every eigenfunction corresponding to an eigenvalue greater than λ_1 has to change sign. As $\varphi_1>0$ in $\overline{\Omega}$ and φ_1 belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $0<\alpha<1$ it follows that $\varphi_1\in\operatorname{int}(C^1(\overline{\Omega})_+)$ where $\operatorname{int}(C^1(\overline{\Omega})_+)$ denotes the interior of the positive cone $C^1(\overline{\Omega})_+=\{u\in C^1(\overline{\Omega}):u(x)\geq 0,\forall x\in\Omega\}$ in the Banach space $C^1(\overline{\Omega})$, which is nonempty and given by

$$\operatorname{int}(C^1(\overline{\Omega})_+)=\left\{u\in C^1(\overline{\Omega}):u(x)>0,\forall x\in\overline{\Omega}\right\}.$$

Remark 2.2. *If $\lambda_1^{(\beta)}$ is the first eigenvalue of the Robin eigenvalue problem (2.4) corresponding to the fixed value $\beta > 0$ and if θ is a real parameter satisfying $0 < \theta < \beta$, then we see from (2.5) that $\lambda_1^{(\beta)} \geq \lambda_1^{(\theta)}$ where $\lambda_1^{(\theta)}$ is the first eigenvalue of (2.4) concerning the value θ. This note is required in the end of the proof of Theorem 4.1 demonstrating the existence of extremal constant-sign solutions of (1.1).*

Let us turn back to the Robin Fučík spectrum which was recently studied in [24] through a variational approach using a mountain-pass procedure. More precisely, it was shown that $\widehat{\Sigma}_p$ contains a first nontrivial curve $\mathcal{C}$ which can be expressed as

$$\mathcal{C} = \{(s + c(s), c(s)), (c(s), s + c(s)) : s \geq 0\}, \tag{2.6}$$

where $c(s)$ is given by

$$c(s) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,+1]} \tilde{J}_s(u),$$

with

$$\Gamma = \{\gamma \in C([-1,1], S) : \gamma(-1) = -\varphi_1 \text{ and } \gamma(1) = \varphi_1\}. \tag{2.7}$$

Here, $\tilde{J}_s$ is equal to the restriction of the C^1-functional $J_s : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J_s(u) = \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\mu - s \int_\Omega (u^+)^p dx$$

to the C^1-submanifold

$$S = \left\{ u \in W^{1,p}(\Omega) : \int_\Omega |u|^p dx = 1 \right\} \tag{2.8}$$

of $W^{1,p}(\Omega)$. This first nontrivial curve stated in (2.6) is Lipschitz continuous, decreasing and its asymptotic behavior can be described by

$$\lim_{s \to +\infty} c(s) = \begin{cases} \lambda_1 & \text{if } p \leq N \\ \overline{\lambda} > \lambda_1 & \text{if } p > N \end{cases}$$

where

$$\overline{\lambda} = \inf_{u \in L} \max_{r \in \mathbb{R}} \frac{\int_\Omega |\nabla(r\varphi_1 + u)|^p dx + \beta \int_{\partial\Omega} |r\varphi_1 + u|^p d\mu}{\int_\Omega |r\varphi_1 + u|^p dx},$$

with

$$L = \{u \in W^{1,p}(\Omega) : u \text{ vanishes somewhere in } \overline{\Omega}, u \not\equiv 0\}$$

(see [24, Proposition 4.2 and Theorem 4.3]). With the help of this first nontrivial curve, we can formulate our last hypothesis on the given data in (1.1).

(H8) Let β be chosen such that $0 < \theta < \beta$ and let $(a, b) \in \mathbb{R}^2_+$ be above the first nontrivial curve $\mathcal{C}$ of the Fučík spectrum $\widehat{\Sigma}_p$ constructed in [24].

In case $a = b = \lambda$ condition (H8) reduces to the assumption that the value λ is strictly greater than λ_2 being the second eigenvalue of the Robin eigenvalue problem of $-\Delta_p$ because of the fact that the point (λ_2, λ_2) belongs to $\widehat{\Sigma}_p$. In [24] a variational characterization of this eigenvalue is obtained by the representation

$$\lambda_2 = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,1]} \left[\int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\sigma \right].$$

For a detailed summary about the Fučík spectrum of the negative p-Laplacian with different boundary conditions we refer to the recent overview article in [25].

A second problem which plays an important part in our treatment is the subsequent boundary value problem

$$-\Delta_p u = -|u|^{p-2}u + 1 \quad \text{in } \Omega, \qquad |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 1 \quad \text{on } \partial\Omega, \tag{2.9}$$

which means that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx = -\int_\Omega (|u|^{p-2}u - 1)v dx + \int_{\partial\Omega} v d\mu \tag{2.10}$$

is fulfilled for every test function $v \in W^{1,p}(\Omega)$. From the classical existence theory we infer the existence of a weak solution of problem (2.9). Testing (2.10) with $v = e_1 - e_2$, where $e_1, e_2 \in W^{1,p}(\Omega)$ are two weak solutions of (2.10), we get the uniqueness. Denote by $e \in W^{1,p}(\Omega)$ the unique weak solution of (2.9), we see at once that e must be nonnegative (testing with $v = e^-$). Further, we obtain $e \in L^\infty(\Omega)$ (see [31, Theorem 4.1] or [33, Corollary 1.2]) and from the regularity results of Lieberman [20] it follows $e \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$. Taking (2.9) into account we have

$$\Delta_p e = |e|^{p-2}e - 1 \le e^{p-1} \quad \text{a.e. in } \Omega.$$

Defining $\beta : [0,\infty) \to \mathbb{R}$ through $\beta(s) = s^{p-1}$ for $s > 0$ we may apply Vázquez's strong maximum principle (see [27, Theorem 5]) to get $e(x) > 0$ for all $x \in \Omega$. Fixing $x_0 \in \partial\Omega$ such that $e(x_0) = 0$ and using again Vázquez's strong maximum principle we conclude that $\partial u/\partial\nu(x_0) < 0$. From the boundary condition in (2.9) we obtain $|\nabla u|^{p-2}\partial u/\partial\nu(x_0) = 1$ which is a contradiction. Hence, $e(x) > 0$ in $\overline{\Omega}$ guaranteeing $e \in \operatorname{int}(C^1(\overline{\Omega})_+)$.

3. Existence of sub- and supersolutions.

In this section we provide the existence of some pairs of weak sub- and supersolutions of our problem (1.1). Here and in the rest of the paper we denote by φ_1 the first eigenfunction of the Robin eigenvalue problem (2.4) corresponding to the first eigenvalue λ_1 related to the fixed parameter β. The function e stands for the unique weak solution of problem (2.9). The main result in this section is the following.

Lemma 3.1. *Let the assumptions in (H) be satisfied and suppose that $a, b > \lambda_1$ as well as $0 < \theta < \beta$. Then there are constants $\vartheta_a, \vartheta_b > 0$ depending on a and b, respectively, such that $\vartheta_a e$ is a positive weak supersolution and $-\vartheta_b e$ is a negative weak subsolution of problem (1.1). Additionally, the function $\varepsilon\varphi_1$ is a positive weak subsolution of problem (1.1) while $-\varepsilon\varphi_1$ is a negative weak supersolution provided the number $\varepsilon > 0$ is sufficiently small.*

Proof. We start to show that $\vartheta_a e$ is a positive weak supersolution of (1.1) with a positive constant ϑ_a to be specified. From (2.10) we obtain

$$\begin{aligned}&\int_\Omega |\nabla(\vartheta_a e)|^{p-2}\nabla(\vartheta_a e)\cdot\nabla v dx \\ &= -\int_\Omega (\vartheta_a e)^{p-1} v dx + \int_\Omega \vartheta_a^{p-1} v dx + \int_{\partial\Omega} \vartheta_a^{p-1} v d\mu, \quad \forall v \in W^{1,p}(\Omega).\end{aligned} \tag{3.1}$$

Combining the definition of a weak supersolution and equation (3.1), we have to show that the inequality

$$\begin{aligned}\int_\Omega (\vartheta_a^{p-1} - (1+a)(\vartheta_a e)^{p-1} - f(x,\vartheta_a e))v dx \\ + \int_{\partial\Omega} (\vartheta_a^{p-1} + \theta(\vartheta_a e)^{p-1} - h(x,\vartheta_a e))v d\mu \geq 0\end{aligned} \tag{3.2}$$

is satisfied for all nonnegative test functions $v \in W^{1,p}(\Omega)$. Thanks to condition (H2) there exists a number $s_a > 0$ such that

$$\frac{f(x,s)}{s^{p-1}} < -(1+a), \quad \text{for a.a. } x \in \Omega \text{ and all } s > s_a. \tag{3.3}$$

With the aid of (H1), one gets

$$f(x,s) + (1+a)s^{p-1} \leq c_a, \quad \text{for a.a. } x \in \Omega \text{ and all } s \in [0, s_a] \tag{3.4}$$

with a constant c_a depending on a. Finally, from (3.3) and (3.4) it follows

$$f(x,s) \leq -(1+a)s^{p-1} + c_a, \quad \text{for a.a. } x \in \Omega \text{ and all } s \geq 0. \tag{3.5}$$

From (H5) we obtain the existence of $s_\theta > 0$ such that

$$\frac{h(x,s)}{s^{p-1}} < \theta, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s > s_\theta,$$

and condition (H4) yields a constant $c_\theta > 0$ such that

$$h(x,s) \leq c_\theta, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \in [0, s_\theta].$$

Consequently this leads to

$$h(x,s) \leq \theta s^{p-1} + c_\theta, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \geq 0. \tag{3.6}$$

Now we can estimate the integrals in (3.2) using the inequalities in (3.5) and (3.6). It results in

$$\begin{aligned}&\int_\Omega (\vartheta_a^{p-1} - (1+a)(\vartheta_a e)^{p-1} - f(x,\vartheta_a e))v dx \\ &\quad + \int_{\partial\Omega} (\vartheta_a^{p-1} + \theta(\vartheta_a e)^{p-1} - h(x,\vartheta_a e))v d\mu \\ &\geq \int_\Omega (\vartheta_a^{p-1} - (1+a)(\vartheta_a e)^{p-1} + (1+a)(\vartheta_a e)^{p-1} - c_a)v dx \\ &\quad + \int_{\partial\Omega} (\vartheta_a^{p-1} + \theta(\vartheta_a e)^{p-1} - \theta(\vartheta_a e)^{p-1} - c_\theta)v d\mu \\ &= \int_\Omega (\vartheta_a^{p-1} - c_a)v dx + \int_{\partial\Omega} (\vartheta_a^{p-1} - c_\theta)v d\mu,\end{aligned}$$

for all $v \in W^{1,p}(\Omega)_+$. From the choice $\vartheta_a := \max\left\{c_a^{\frac{1}{p-1}}, c_\theta^{\frac{1}{p-1}}\right\}$ we conclude that the function $\overline{u} = \vartheta_a e$ is a positive weak supersolution of our problem (1.1). Following the same pattern one can prove that $\underline{u} = -\vartheta_b e$ is a negative weak subsolution of (1.1).

Let us prove the second part of the lemma. To this end, we consider the weak formulation of the Robin eigenvalue problem of the p-Laplacian multiplied with the

parameter $\varepsilon^{p-1} > 0$, namely

$$\begin{aligned}&\int_\Omega |\nabla(\varepsilon\varphi_1)|^{p-2}\nabla(\varepsilon\varphi_1)\cdot\nabla v dx\\ &= \int_\Omega \lambda_1(\varepsilon\varphi_1)^{p-1}v dx - \int_{\partial\Omega}\beta(\varepsilon\varphi_1)^{p-1}v d\mu, \quad \forall v\in W^{1,p}(\Omega).\end{aligned}$$

Taking the definition of a weak subsolution into account we have to prove that

$$\begin{aligned}&\int_\Omega ((\lambda_1 - a)(\varepsilon\varphi_1)^{p-1} - f(x,\varepsilon\varphi_1))v dx\\ &\quad + \int_{\partial\Omega}((\theta-\beta)(\varepsilon\varphi_1)^{p-1} - h(x,\varepsilon\varphi_1))v d\mu \le 0\end{aligned} \tag{3.7}$$

is fulfilled for all $v \in W^{1,p}(\Omega)_+$. Applying the assumptions (H3) and (H6) provides the existence of two numbers $\delta_a > 0$ and $\delta_\theta > 0$ such that

$$\begin{aligned}&\frac{|f(x,s)|}{|s|^{p-1}} < a - \lambda_1, \quad \text{for a.a. } x\in\Omega \text{ and all } 0<|s|\le\delta_a,\\ &\frac{|h(x,s)|}{|s|^{p-1}} < \beta - \theta, \quad \text{for a.a. } x\in\partial\Omega \text{ and all } 0<|s|\le\delta_\theta,\end{aligned} \tag{3.8}$$

due to the fact that $a > \lambda_1$ and $\beta > \theta$. Choosing

$$0 < \varepsilon \le \min\left\{\frac{\delta_a}{\|\varphi_1\|_\infty}, \frac{\delta_\theta}{\|\varphi_1\|_\infty}\right\},$$

where $\|\varphi_1\|_\infty$ stands for the supremum-norm of φ_1, along with (3.8), we obtain from (3.7)

$$\begin{aligned}&\int_\Omega ((\lambda_1 - a)(\varepsilon\varphi_1)^{p-1} - f(x,\varepsilon\varphi_1))v dx\\ &\quad + \int_{\partial\Omega}((\theta-\beta)(\varepsilon\varphi_1)^{p-1} - h(x,\varepsilon\varphi_1))v d\mu\\ &\le \int_\Omega ((\lambda_1 - a)(\varepsilon\varphi_1)^{p-1} + (a-\lambda_1)(\varepsilon\varphi_1)^{p-1})v dx\\ &\quad + \int_{\partial\Omega}((\theta-\beta)(\varepsilon\varphi_1)^{p-1} + (\beta-\theta)(\varepsilon\varphi_1)^{p-1})v d\mu\\ &= 0,\end{aligned}$$

which proves the assertion. The existence of a negative weak supersolution $-\varepsilon\varphi_1$ can be shown in a similar way. □

Remark 3.2. *Note that every nontrivial weak solution $u \in [0, \vartheta_a e]$ of (1.1) belongs to $\operatorname{int}(C^1(\overline{\Omega})_+)$. This follows from the $C^{1,\alpha}$-regularity of Lieberman [20] combined with Vázquez's strong maximum principle [27] and the growth properties of f and h given in Corollary 2.1. The same holds true for every nontrivial weak solution $u \in [-\vartheta_b e, 0]$ meaning that u lies in $-\operatorname{int}(C^1(\overline{\Omega})_+)$.*

4. **Extremal constant-sign solutions.** The main result in this section is the following theorem about the existence of extremal constant-sign solutions of problem (1.1).

Theorem 4.1. *Let the conditions in (H) be satisfied and let $0 < \theta < \beta$. Then, for every $a > \lambda_1$ and $b \in \mathbb{R}$, there exists a smallest positive weak solution $u_+ = u_+(a) \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of (1.1) in the order interval $[0, \vartheta_a e]$ while for every $b > \lambda_1$ and $a \in \mathbb{R}$, there exists a greatest negative weak solution $u_- = u_-(b) \in -\operatorname{int}(C^1(\overline{\Omega})_+)$ within $[-\vartheta_b e, 0]$.*

Proof. We only prove the assertion for the smallest positive weak solution, the other case acts in the same way. From Lemma 3.1 we know the existence of a positive weak subsolution $\varepsilon\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and a positive weak supersolution $\vartheta_a e \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Taking $\varepsilon > 0$ small enough such that $\varepsilon\varphi_1 \leq \vartheta_a e$ provides an ordered pair of weak sub- and supersolutions of problem (1.1), namely $[\varepsilon\varphi_1, \vartheta_a e]$. The method of weak sub- and supersolution concerning problems of type (1.1) (see [7]) ensures the existence of a smallest positive weak solution $u_\varepsilon = u_\varepsilon(a)$ of (1.1) lying between $\varepsilon\varphi_1$ and $\vartheta_a e$. Taking into account Remark 3.2 we obtain that $u_\varepsilon \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Therefore, for every positive integer n sufficiently large, there exists a smallest positive weak solution $u_n \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of problem (1.1) satisfying $\frac{1}{n}\varphi_1 \leq u_n \leq \vartheta_a e$. From this we get a sequence (u_n) of smallest positive weak solutions being monotone decreasing. It follows

$$u_n \downarrow u_+ \quad \text{pointwise} \tag{4.1}$$

with a function $u_+ : \Omega \to \mathbb{R}$ belonging to $[0, \vartheta_a e]$.

Let us show that u_+ solves problem (1.1). As $u_n \in [\frac{1}{n}\varphi_1, \vartheta_a e]$, one can easily prove the boundedness of (u_n) in $W^{1,p}(\Omega)$. Thus, there is a weakly convergent subsequence of (u_n) and due to the monotonicity of (u_n) along with the compact embeddings $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ as well as $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, the entire sequence (u_n) has the following convergence properties:

$$\begin{aligned} &u_n \rightharpoonup u_+ \quad \text{in } W^{1,p}(\Omega), \\ &u_n \to u_+ \quad \text{in } L^p(\Omega), \text{ in } L^p(\partial\Omega), \text{ for a.a. } x \in \Omega, \text{ and for a.a. } x \in \partial\Omega. \end{aligned} \tag{4.2}$$

As u_n solves problem (1.1), we have

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dx \\ &= \int_\Omega (a u_n^{p-1} + f(x, u_n))v dx + \int_{\partial\Omega} (h(x, u_n) - \theta u_n^{p-1})v d\mu, \end{aligned} \tag{4.3}$$

for all $v \in W^{1,p}(\Omega)$. Taking the test function $v = u_n - u_+ \in W^{1,p}(\Omega)$ leads to

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u_+) dx \\ &= \int_\Omega (a u_n^{p-1} + f(x, u_n))(u_n - u_+) dx + \int_{\partial\Omega} (h(x, u_n) - \theta u_n^{p-1})(u_n - u_+) d\mu. \end{aligned}$$

Thanks to the boundedness of f and h in combination with the convergence properties in (4.2) and the uniform boundedness of the sequence (u_n), we get by applying Lebesgue's dominated convergence that

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u_+) dx = 0,$$

which by the (S_+)-property of $-\Delta_p$ on $W^{1,p}(\Omega)$ implies

$$u_n \to u_+ \quad \text{in } W^{1,p}(\Omega). \tag{4.4}$$

The strong convergence in (4.4) along with (H1), (H4) and the uniform boundedness of (u_n) allows us to pass to the limit in (4.3) which ensures that u_+ is in fact a weak solution of (1.1).

Taking into account Remark 3.2 we know that $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$ provided $u \not\equiv 0$. Arguing by contradiction, we suppose that $u_+ \equiv 0$ implying that (see (4.1))

$$u_n(x) \downarrow 0 \quad \text{for all } x \in \Omega. \tag{4.5}$$

Setting

$$w_n := \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}} \quad \text{for all } n,$$

we may suppose that, along a subsequence denoted again by w_n,

$$\begin{aligned} &w_n \rightharpoonup w \quad \text{in } W^{1,p}(\Omega), \\ &w_n \to w \quad \text{in } L^p(\Omega), \text{ in } L^p(\partial\Omega), \text{ for a.a. } x \in \Omega, \text{ for a.a. } x \in \partial\Omega, \end{aligned} \tag{4.6}$$

with some function $w \in W^{1,p}(\Omega)$. Additionally, there exist functions $k_1 \in L^p(\Omega)_+$, $k_2 \in L^p(\partial\Omega)_+$ such that

$$\begin{aligned} &|w_n(x)| \le k_1(x) \quad \text{for a.a. all } x \in \Omega, \\ &|w_n(x)| \le k_2(x) \quad \text{for a.a. all } x \in \partial\Omega. \end{aligned} \tag{4.7}$$

Using the representation $u_n = \|u_n\|_{W^{1,p}(\Omega)} w_n$ we have from (4.3) the variational equation

$$\begin{aligned} &\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \cdot \nabla v dx \\ &= \int_\Omega \left(a w_n^{p-1} + \frac{f(x,u_n)}{u_n^{p-1}} w_n^{p-1} \right) v dx + \int_{\partial\Omega} \left(\frac{h(x,u_n)}{u_n^{p-1}} w_n^{p-1} - \theta w_n^{p-1} \right) v d\mu, \end{aligned} \tag{4.8}$$

for all $v \in W^{1,p}(\Omega)$. Particularly, for the choice $v = w_n - w \in W^{1,p}(\Omega)$, one gets

$$\begin{aligned} &\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \cdot \nabla (w_n - w) dx \\ &= \int_\Omega \left(a w_n^{p-1} + \frac{f(x,u_n)}{u_n^{p-1}} w_n^{p-1} \right) (w_n - w) dx \\ &+ \int_{\partial\Omega} \left(\frac{h(x,u_n)}{u_n^{p-1}} w_n^{p-1} - \theta w_n^{p-1} \right) (w_n - w) d\mu. \end{aligned} \tag{4.9}$$

Applying Corollary 2.1 with $\xi = \vartheta_a \|e\|_\infty$ there exist constants $c_f, c_h > 0$ such that

$$\begin{aligned} &\frac{|f(x,u_n(x))|}{u_n^{p-1}(x)} w_n^{p-1}(x)|w_n(x) - w(x)| \le c_f k_1(x)^{p-1}(k_1(x) + |w(x)|), \\ &\frac{|h(x,u_n(x))|}{u_n^{p-1}(x)} w_n^{p-1}(x)|w_n(x) - w(x)| \le c_h k_2(x)^{p-1}(k_2(x) + |w(x)|), \end{aligned} \tag{4.10}$$

where (4.7) is also taken into account. As the right-hand sides of (4.10) are in $L^1(\Omega)$ and $L^1(\partial\Omega)$, respectively, we may apply Lebesgue's dominated convergence theorem, which associated with (4.6) provides

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)}{u_n^{p-1}} w_n^{p-1}(w_n - w) dx = 0, \\ &\lim_{n\to\infty} \int_{\partial\Omega} \frac{h(x,u_n)}{u_n^{p-1}} w_n^{p-1}(w_n - w) d\mu = 0. \end{aligned} \tag{4.11}$$

From (4.9) in conjunction with (4.11) we derive

$$\limsup_{n\to\infty}\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\cdot\nabla(w_n-w)dx=0.$$

Applying again the (S_+)-property of $-\Delta_p$ corresponding to $W^{1,p}(\Omega)$ yields

$$w_n\to w \quad \text{in } W^{1,p}(\Omega), \tag{4.12}$$

while $\|w\|_{W^{1,p}(\Omega)}=1$ meaning $w\not\equiv 0$. Taking into account (4.5), (4.12), and the assumptions (H3) and (H6), we may pass to the limit in (4.8) which results in

$$\int_\Omega |\nabla w|^{p-2}\nabla w\cdot\nabla v dx=a\int_\Omega w^{p-1}vdx-\theta\int_{\partial\Omega}w^{p-1}vd\mu,\quad \forall v\in W^{1,p}(\Omega). \tag{4.13}$$

Since $w\not\equiv 0$, equation (4.13) represents the Robin eigenvalue problem of the negative p-Laplacian $-\Delta_p$ with the eigenfunction $w\geq 0$ corresponding to the eigenvalue a and related to the parameter θ. By means of Remark 2.2 and due to the assumptions $a>\lambda_1$ and $0<\theta<\beta$, we see that a is also greater than the first eigenvalue of the Robin eigenvalue problem corresponding to the positive number θ. However, this contradicts the results of Lê [18] because w must change sign in Ω. Hence, $u_+\not\equiv 0$ concluding $u_+\in \operatorname{int}(C^1(\overline{\Omega})_+)$.

Finally, we have to check that u_+ is indeed the smallest positive weak solution in $[0,\vartheta_a e]$. To this end, let $u\in W^{1,p}(\Omega), 0\leq u\leq \vartheta_a e, u\not\equiv 0$ be a weak solution of (1.1). Remark 3.2 ensures that $u\in \operatorname{int}(C^1(\overline{\Omega})_+)$. This implies the existence of an integer n sufficiently large such that $u\in[\frac{1}{n}\varphi_1,\vartheta_a e]$. As we already know, u_n is the smallest weak solution in the ordered interval $[\frac{1}{n}\varphi_1,\vartheta_a e]$ meaning that $u_n\leq u$. Making use of (4.1), we get $u_+\leq u$ which proves that $u_+\in\operatorname{int}(C^1(\overline{\Omega})_+)$ is the smallest weak solution of problem (1.1) within $[0,\vartheta_a e]$. □

Remark 4.2. *Regarding Theorem 4.1 the next proceeding is to find a third nontrivial weak solution u_0 which lies between u_- and u_+. If $u_0\neq u_-$ and $u_0\neq u_+$, then it must be a sign-changing weak solution of (1.1) due to the extremality of u_+ and u_-.*

5. **Sign-changing solution.** In this section we prove the existence of a nontrivial sign-changing weak solution $u_0\in C^1(\overline{\Omega})$ of (1.1) which belongs to the ordered interval $[u_-,u_+]$.

To this end, let $\tau_+,\tau_-,\tau_0:\Omega\times\mathbb{R}\to\mathbb{R}$ be truncation operators defined as:

$$\tau_+(x,s)=\begin{cases}0 & \text{if } s<0\\ s & \text{if } 0\leq s\leq u_+(x)\\ u_+(x) & \text{if } s>u_+(x)\end{cases},\quad \tau_-(x,s)=\begin{cases}u_-(x) & \text{if } s<u_-(x)\\ s & \text{if } u_-(x)\leq s\leq 0\\ 0 & \text{if } s>0\end{cases},$$

$$\tau_0(x,s)=\begin{cases}u_-(x) & \text{if } s<u_-(x)\\ s & \text{if } u_-(x)\leq s\leq u_+(x)\\ u_+(x) & \text{if } s>u_+(x)\end{cases}.$$

Further, we denote by $\tau_+^{\partial\Omega},\tau_-^{\partial\Omega},\tau_0^{\partial\Omega}:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ the corresponding truncation operators defined on $\partial\Omega$. We see at once that these truncation functions are continuous, uniformly bounded, and even Lipschitz continuous with respect to the second

argument. Taking into account these truncations we introduce the subsequent associated functionals through

$$\begin{aligned}
J_+(u) =& \frac{1}{p}\left(\|\nabla u\|^p_{L^p(\Omega)} + \theta\|u\|^p_{L^p(\partial\Omega)}\right)\\
&- \int_\Omega \int_0^{u(x)} \left(a\tau_+(x,s)^{p-1} + f(x,\tau_+(x,s))\right) dsdx\\
&- \int_{\partial\Omega} \int_0^{u(x)} h(x,\tau_+^{\partial\Omega}(x,s))dsd\mu,\\
J_-(u) =& \frac{1}{p}\left(\|\nabla u\|^p_{L^p(\Omega)} + \theta\|u\|^p_{L^p(\partial\Omega)}\right)\\
&- \int_\Omega \int_0^{u(x)} \left(-b|\tau_-(x,s)|^{p-1} + f(x,\tau_-(x,s))\right) dsdx\\
&- \int_{\partial\Omega} \int_0^{u(x)} h(x,\tau_-^{\partial\Omega}(x,s))dsd\mu,\\
J_0(u) =& \frac{1}{p}\left(\|\nabla u\|^p_{L^p(\Omega)} + \theta\|u\|^p_{L^p(\partial\Omega)}\right)\\
&- \int_\Omega \int_0^{u(x)} \left(a\tau_+(x,s)^{p-1} - b|\tau_-(x,s)|^{p-1} + f(x,\tau_0(x,s))\right) dsdx\\
&- \int_{\partial\Omega} \int_0^{u(x)} h(x,\tau_0^{\partial\Omega}(x,s))dsd\mu.
\end{aligned}$$

These functionals are well-defined and differentiable. Thanks to the truncation operators combined with the equivalent norm stated in (2.1) (replacing ζ by θ) it can be shown that $J_-, J_+, J_0 : W^{1,p}(\Omega) \to \mathbb{R}$ are coercive and weakly sequentially lower semicontinuous implying the existence of their global minimizers (cf. e.g. [34, Proposition 38.15]). Further, the functionals fulfill the classical Palais-Smale condition. A characterization of their critical points is stated in the next lemma.

Lemma 5.1. *Every critical point $\omega \in W^{1,p}(\Omega)$ of $J_+(J_-)$ is a nonnegative (nonpositive) weak solution of (1.1) such that $0 \le \omega \le u_+$ ($u_- \le \omega \le 0$), where u_+ and u_- denote the extremal constant-sign solutions of (1.1) obtained in Theorem 4.1. If $\omega \in W^{1,p}(\Omega)$ is a critical point of J_0, then ω is a weak solution of (1.1) satisfying $u_- \le \omega \le u_+$.*

Proof. Let us show the last assertion, the other ones can be done similarly. Suppose $\omega \in W^{1,p}(\Omega)$ is a critical point of J_0, then it holds $J_0'(\omega) = 0$ meaning that

$$\begin{aligned}
&\int_\Omega |\nabla\omega|^{p-2}\nabla\omega\cdot\nabla v dx + \theta\int_{\partial\Omega}|\omega|^{p-2}\omega v d\mu\\
&= \int_\Omega \left(a\tau_+(x,\omega)^{p-1} - b|\tau_-(x,\omega)|^{p-1} + f(x,\tau_0(x,\omega))\right) v dx\\
&\quad + \int_{\partial\Omega} h(x,\tau_0^{\partial\Omega}(x,\omega))v d\mu, \quad \forall v \in W^{1,p}(\Omega).
\end{aligned}$$

Since u_+ is a positive weak solution of (1.1), we obtain

$$\begin{aligned}&\int_\Omega |\nabla u_+|^{p-2}\nabla u_+ \cdot \nabla v dx\\ &= \int_\Omega \left(a u_+^{p-1} + f(x,u_+)\right) v dx + \int_{\partial\Omega} \left(h(x,u_+) - \theta u_+^{p-1}\right) v d\mu,\end{aligned}$$

for all $v \in W^{1,p}(\Omega)$. Now, putting $v = (\omega - u_+)^+$ and combining both equations above, one has

$$\begin{aligned}&\int_\Omega \left(|\nabla\omega|^{p-2}\nabla\omega - |\nabla u_+|^{p-2}\nabla u_+\right)\cdot\nabla(\omega-u_+)^+ dx\\ &\quad + \theta\int_{\partial\Omega}\left(|\omega|^{p-2}\omega - u_+^{p-1}\right)(\omega-u_+)^+ d\mu\\ &= \int_\Omega \left(a\tau_+(x,\omega)^{p-1} - b|\tau_-(x,\omega)|^{p-1} - a u_+^{p-1}\right)(\omega-u_+)^+ dx\\ &\quad + \int_\Omega \left(f(x,\tau_0(x,\omega)) - f(x,u_+)\right)(\omega-u_+)^+ dx\\ &\quad + \int_{\partial\Omega}\left(h(x,\tau_0^{\partial\Omega}(x,s)) - h(x,u_+)\right)(\omega-u_+)^+ d\mu.\end{aligned} \tag{5.1}$$

With a view to the definition of the truncation operators it is easy to see that the right-hand side of (5.1) vanishes. However, if $\omega > u_+$, the left-hand side is strictly positive (cf. e.g. [8, p. 37]). Hence, it must hold $\omega \le u_+$. In order to prove $u_- \le \omega$ we can proceed in the same line which yields $u_- \le \omega \le u_+$. Taking again the definition of the truncations into account we have $\tau_+(x,\omega) = \omega^+, |\tau_-(x,\omega)| = \omega^-$, $\tau_0(x,\omega) = \omega$ and $\tau_0^{\partial\Omega}(x,\omega) = \omega$ meaning that ω is a weak solution of (1.1) satisfying $u_- \le \omega \le u_+$. □

Lemma 5.2. *Suppose (H) and let $a, b > \lambda_1$ and $\beta > \theta > 0$. Then the extremal positive (negative) weak solution u_+ (u_-) of (1.1) is the unique global minimizer of the functional J_+ (J_-) while both of them are local minimizers of the functional J_0 as well. Further, J_0 possesses a global minimizer ω_0 being a nontrivial weak solution of (1.1) satisfying $u_- \le \omega_0 \le u_+$.*

Proof. Let $\omega_+ \in W^{1,p}(\Omega)$ the global minimizer of J_+ which exists due to the property of J_+ to be coercive and weakly sequentially lower semicontinuous. Concerning Lemma 5.1 the critical point ω_+ is a nonnegative weak solution to equation (1.1) belonging to $[0, u_+]$. In order to verify that ω_+ is unequal zero, we have to show that $J_+(\omega_+) \neq 0$. According to hypothesis (H3) and (H6) we find numbers $\delta_a > 0$ and $\delta_\theta > 0$ such that

$$\begin{aligned}|f(x,s)| &\le (a-\lambda_1)s^{p-1}, \quad \forall s : 0 < s \le \delta_a,\\ |h(x,s)| &\le (\beta-\theta)s^{p-1}, \quad \forall s : 0 < s \le \delta_\theta,\end{aligned} \tag{5.2}$$

since $a > \lambda_1$ and $\beta > \theta$. We put $\varepsilon > 0$ sufficiently small such that

$$\varepsilon\varphi_1 < u_+, \quad \varepsilon\|\varphi_1\|_\infty < \delta_a, \quad \varepsilon\|\varphi_1\|_\infty < \delta_\theta.$$

From (5.2) combined with the Robin eigenvalue problem we infer

$$\begin{aligned} J_+(\varepsilon\varphi_1) &= \frac{\lambda_1 - a}{p}\varepsilon^p \|\varphi_1\|^p_{L^p(\Omega)} \\ &\quad - \int_\Omega \int_0^{\varepsilon\varphi_1(x)} f(x,s)dsdx + \frac{\theta-\beta}{p}\varepsilon^p \|\varphi_1\|^p_{L^p(\partial\Omega)} - \int_{\partial\Omega}\int_0^{\varepsilon\varphi_1(x)} h(x,s)ds d\mu \\ &< \frac{\lambda_1 - a}{p}\varepsilon^p \|\varphi_1\|^p_{L^p(\Omega)} + \int_\Omega \int_0^{\varepsilon\varphi_1(x)} (a-\lambda_1)s^{p-1}dsdx + \frac{\theta-\beta}{p}\varepsilon^p \|\varphi_1\|^p_{L^p(\partial\Omega)} \\ &\quad + \int_{\partial\Omega}\int_0^{\varepsilon\varphi_1(x)} (\beta-\theta)s^{p-1}ds d\mu \\ &= 0. \end{aligned}$$

Hence, $J_+(\omega_+) < 0$ meaning that $\omega_+ \not\equiv 0$. This yields that $\omega_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$ (cf. Remark 3.2). Due to the fact that u_+ is the smallest positive weak solution of (1.1) in $[0, \vartheta_a e]$ satisfying $0 \leq \omega_+ \leq u_+$ we obtain $\omega_+ = u_+$ proving that u_+ must be the unique global minimizer of J_+. In a similar way, we get that u_- is the unique global minimizer of J_-. As $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$, there exists a neighborhood V_{u_+} of u_+ in the space $C^1(\overline{\Omega})$ satisfying $V_{u_+} \subset C^1(\overline{\Omega})_+$. Hence, it holds $J_+ = J_0$ on V_{u_+} meaning that u_+ is a local minimizer of J_0 on $C^1(\overline{\Omega})$. From [30] we know that u_+ is a local minimizer on $W^{1,p}(\Omega)$ as well. Similarly, we obtain that u_- is a local minimizer of J_0 with respect to $W^{1,p}(\Omega)$.

As mentioned at the beginning of this section the functional $J_0 : W^{1,p}(\Omega) \to \mathbb{R}$ is coercive and weakly sequentially lower semicontinuous. That means that its global minimizer, namely ω_0, exists. Taking into account Lemma 5.1 we get that the critical point ω_0 is a solution of (1.1) satisfying $u_- \leq \omega_0 \leq u_+$. Since $J_0(u_+) = J_+(u_+) < 0$ it follows that ω_0 must be nontrivial meaning $\omega_0 \not\equiv 0$. □

Now, we are in the position to prove the main result in this section.

Theorem 5.3. *Under the hypotheses (H) and (H8) problem (1.1) possesses a nontrivial sign-changing weak solution $u_0 \in C^1(\overline{\Omega})$.*

Proof. In Lemma 5.2 it has been shown that the functional $J_0 : W^{1,p}(\Omega) \to \mathbb{R}$ possesses a global minimizer $\omega_0 \in W^{1,p}(\Omega)$ which is a nontrivial weak solution of our original problem (1.1) lying between u_- and u_+. If $\omega_0 \neq u_-$ and $\omega_0 \neq u_+$, then $u_0 = \omega_0$ must be a sign-changing weak solution of (1.1) due to the extremality properties of the constant-sign solutions u_- and u_+ (cf. Theorem 4.1). In this case we are done.

Let us prove the assertion if either $\omega_0 = u_-$ or $\omega_0 = u_+$ is satisfied. We only show the case $\omega_0 = u_+$, the other one can be done likewise. From Lemma 5.2 it is known that u_- is a local minimizer of J_0. Without loss of generality we can assume that u_- is a strict local minimizer of J_0 elsewise there would exist infinitely many critical points ω of J_0 being sign-changing weak solutions of (1.1) because of the relation $u_- \leq \omega \leq u_+$ combined with the fact that u_- as well as u_+ are extremal constant-sign solutions. With the aid of these assumptions we find a number $\rho \in \big(0, \|u_+ - u_-\|_{W^{1,p}(\Omega)}\big)$ such that

$$J_0(u_+) \leq J_0(u_-) < \inf\{J_0(u) : u \in \partial B_\rho(u_-)\} \tag{5.3}$$

with $\partial B_\rho = \{u \in W^{1,p}(\Omega) : \|u - u_-\|_{W^{1,p}(\Omega)} = \rho\}$. Now, we are able to apply the mountain-pass theorem used to the functional J_0 (see [26] or [28, Theorem 2.4.4]).

Note that J_0 fulfills the Palais-Smale condition which is required at this point. We obtain the existence of a critical point $u_0 \in W^{1,p}(\Omega)$ of J_0, that is $J_0'(u_0) = 0$, satisfying

$$\inf\{J_0(u) : u \in \partial B_\rho(u_-)\} \leq J_0(u_0) = \inf_{\gamma\in\Gamma_W} \max_{t\in[-1,1]} J_0(\gamma(t)), \tag{5.4}$$

where

$$\Gamma_W = \{\gamma \in C([-1,1], W^{1,p}(\Omega)) : \gamma(-1) = u_-, \gamma(1) = u_+\}.$$

We easily see from (5.3) and (5.4) that u_0 can not be u_- as well as u_+. Therefore, the critical point u_0 is a nontrivial sign-changing weak solution of (1.1) provided $u_0 \neq 0$. That means we have to prove that $J_0(u_0) \neq 0$, which is satisfied if there exists a path $\widetilde{\gamma} \in \Gamma_W$ such that

$$J_0(\widetilde{\gamma}(t)) \neq 0, \quad \text{for all } t \in [-1,1]. \tag{5.5}$$

Let S (defined in (2.8)) and $S_C = S \cap C^1(\overline{\Omega})$ be equipped with the topologies induced by $W^{1,p}(\Omega)$ and $C^1(\overline{\Omega})$, respectively. We set

$$\Gamma_C = \{\gamma \in C([-1,1], S_C) : \gamma(-1) = -\varphi_1, \gamma(1) = \varphi_1\}$$

while Γ is stated in (2.7). Taking the recent results in [24] into account there exists a continuous path $\gamma \in \Gamma$ satisfying $t \mapsto \gamma(t) \in \{u \in W^{1,p}(\Omega) : J_{(a,b)}(u) < 0, \|u\|_{L^p(\Omega)} = 1\}$ provided the pair (a,b) is above the curve $\mathcal{C}$ of hypothesis (H8). Here, the functional $J_{(a,b)} : W^{1,p}(\Omega) \to \mathbb{R}$ is defined as the potential associated to the Robin Fučík spectrum given by

$$J_{(a,b)}(u) = \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\mu - \int_\Omega (a(u^+)^p + b(u^-)^p) dx.$$

Thanks to this first nontrivial curve $\mathcal{C}$, we find a number $\kappa > 0$ such that

$$J_{(a,b)}(\gamma(t)) \leq -\kappa < 0, \quad \text{for all } t \in [-1,1].$$

Since S_C is dense in S we deduce the density of Γ_C in Γ (for the proof we refer to [28]) which implies the existence of a continuous path $\gamma_C \in \Gamma_C$ such that

$$|J_{(a,b)}(\gamma(t)) - J_{(a,b)}(\gamma_C(t))| < \frac{\kappa}{2}, \quad \text{for all } t \in [-1,1].$$

Further, as the set $\gamma_C([-1,1])(\overline{\Omega})$ is uniformly bounded in $\mathbb{R}$ there exists a constant $M > 0$ such that

$$|\gamma_C(t)(x)| \leq M \quad \text{for all } x \in \overline{\Omega} \text{ and for all } t \in [-1,1].$$

Recall that $u_+, -u_- \in \operatorname{int}(C^1(\overline{\Omega})_+)$ (see Theorem 4.1). Then, for every $u \in \gamma_C([-1,1])$ and any bounded neighborhood U_u of u in $C^1(\overline{\Omega})$, we find positive numbers ς_u and ι_u satisfying

$$u_+ - \varsigma w \in \operatorname{int}(C^1(\overline{\Omega})_+) \quad \text{and} \quad -u_- + \iota w \in \operatorname{int}(C^1(\overline{\Omega})_+), \tag{5.6}$$

for all $\varsigma \in [0, \varsigma_u]$, for all $\iota \in [0, \iota_u]$, and for all $w \in U_u$. With the aid of (5.6) together with a compactness argument we obtain the existence of a number $\varepsilon_C > 0$ such that

$$u_-(x) \leq \varepsilon\gamma_C(t)(x) \leq u_+(x), \tag{5.7}$$

for all $x \in \Omega$, for all $t \in [-1,1]$, and for all $\varepsilon \in (0, \varepsilon_C]$. By means of the representation of $J_{(a,b)}$, we may write the functional J_0 in the subsequent form

$$\begin{aligned} J_0(u) =& \frac{1}{p} J_{(a,b)}(u) + \frac{\theta-\beta}{p}\|u\|^p_{L^p(\partial\Omega)} + \frac{1}{p}\int_\Omega \left(a(u^+)^p + b(u^-)^p\right) dx \\ &- \int_\Omega \int_0^{u(x)} \left(a\tau_+(x,s)^{p-1} - b|\tau_-(x,s)|^{p-1} + f(x,\tau_0(x,s))\right) dsdx \\ &- \int_{\partial\Omega}\int_0^{u(x)} h(x,\tau_0^{\partial\Omega}(x,s))dsd\mu. \end{aligned} \tag{5.8}$$

Applying (5.7) to the new representation of J_0 given in (5.8) it follows, for all $\varepsilon \in (0,\varepsilon_C]$ and all $t \in [-1,1]$,

$$\begin{aligned} J_0(\varepsilon\gamma_C(t)) &< \frac{1}{p} J_{(a,b)}(\varepsilon\gamma_C(t)) - \int_\Omega \int_0^{\varepsilon\gamma_C(t)(x)} f(x,s)dsdx \\ &\quad - \int_{\partial\Omega}\int_0^{\varepsilon\gamma_C(t)(x)} h(x,s)dsd\mu \\ &= \varepsilon^p \left[\frac{1}{p} J_{(a,b)}(\gamma_C(t)) - \frac{1}{\varepsilon^p}\int_\Omega\int_0^{\varepsilon\gamma_C(t)(x)} f(x,s)dsdx \right. \\ &\qquad \left. - \frac{1}{\varepsilon^p}\int_{\partial\Omega}\int_0^{\varepsilon\gamma_C(t)(x)} h(x,s)dsd\mu\right] \\ &< \varepsilon^p \left[-\frac{\kappa}{2p} + \frac{1}{\varepsilon^p}\int_\Omega \left|\int_0^{\varepsilon\gamma_C(t)(x)} f(x,s)ds\right| dx \right. \\ &\qquad \left. + \frac{1}{\varepsilon^p}\int_{\partial\Omega}\left|\int_0^{\varepsilon\gamma_C(t)(x)} h(x,s)ds\right| d\mu\right], \end{aligned} \tag{5.9}$$

where we have used the fact that $\theta < \beta$. Due to the assumptions (H3) and (H6) there are constants $\psi_1 > 0$ and $\psi_2 > 0$ such that

$$\begin{aligned} |f(x,s)| &\le \frac{\kappa}{5M^p|\Omega|}|s|^{p-1}, \quad \text{for a.a. } x \in \Omega \text{ and all } s : |s| \le \psi_1, \\ |h(x,s)| &\le \frac{\kappa}{5M^p|\partial\Omega|}|s|^{p-1}, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s : |s| \le \psi_2. \end{aligned}$$

Now, we choose $\varepsilon > 0$ sufficiently small such that $\varepsilon < \min\left\{\varepsilon_C, \frac{\psi_1}{M}, \frac{\psi_2}{M}\right\}$ to obtain

$$\begin{aligned} \frac{1}{\varepsilon^p}\int_\Omega \left|\int_0^{\varepsilon\gamma_C(t)(x)} f(x,s)ds\right| dx &\le \frac{\kappa}{5p}, \\ \frac{1}{\varepsilon^p}\int_{\partial\Omega}\left|\int_0^{\varepsilon\gamma_C(t)(x)} h(x,s)ds\right| d\mu &\le \frac{\kappa}{5p}. \end{aligned} \tag{5.10}$$

Then, we get from (5.9) combined with (5.10)

$$J_0(\varepsilon\gamma_C(t)) \le \varepsilon^p\left(-\frac{\kappa}{2p} + \frac{\kappa}{5p} + \frac{\kappa}{5p}\right) < 0, \quad \text{for all } t \in [-1,1]. \tag{5.11}$$

Relation (5.11) demonstrates the existence of a continuous path $\varepsilon\gamma_C$ connecting $-\varepsilon\varphi_1$ and $\varepsilon\varphi_1$. In order to prove the latter in (5.5) we have to construct two other paths which shall join $\varepsilon\varphi_1$ and u_+, respectively, u_- and $-\varepsilon\varphi_1$.

As proved in Lemma 5.2 the smallest positive weak solution u_+ is the unique global minimizer of J_+, so we can suppose that $J_+(u_+) < J_+(\varepsilon\varphi_1)$. Further, from Lemma 5.1 it is known that the functional J_+ has no critical values in the interval $(J_+(u_+), J_+(\varepsilon\varphi_1)]$. Since the functional J_+ satisfies the Palais-Smale condition due to its coercivity, the second deformation lemma (cf. [16]) can be applied to J_+. Denote

$$J_+^{\varepsilon\varphi_1} := \{u \in W^{1,p}(\Omega) : J_+(u) \leq J_+(\varepsilon\varphi_1)\},$$

we obtain the existence of a continuous mapping

$$\eta : [0,1] \times J_+^{\varepsilon\varphi_1} \to J_+^{\varepsilon\varphi_1}$$

characterized through

$$(1) \quad \eta(0,u) = u, \qquad (2) \quad \eta(1,u) = u_+, \qquad (3) \quad J_+(\eta(t,u)) \leq J_+(u),$$

for all $t \in [0,1]$ and for all $u \in J_+^{\varepsilon\varphi_1}$. Now we denote by γ_+ a path from $[0,1]$ to $W^{1,p}(\Omega)$ defined by $\gamma_+(t) = \eta(t, \varepsilon\varphi_1)^+ = \max\{\eta(t,\varepsilon\varphi_1), 0\}$ for all $t \in [0,1]$. Clearly, γ_+ is continuous and joins $\varepsilon\varphi_1$ and u_+. Moreover, it satisfies

$$J_0(\gamma_+(t)) = J_+(\gamma_+(t)) \leq J_+(\eta(t,\varepsilon\varphi_1)) \leq J_+(\varepsilon\varphi_1) < 0,$$

for all $t \in [0,1]$. Finally, we may apply the second deformation lemma to the functional J_- making use of the same arguments. We obtain a continuous path $\gamma_- : [0,1] \to W^{1,p}(\Omega)$ connecting $-\varepsilon\varphi_1$ and u_- and satisfying

$$J_0(\gamma_-(t)) < 0, \qquad \text{for all } t \in [0,1].$$

Now, the proof is almost finished. If we put the paths γ_-, γ_C and γ_+ together, we get a continuous path $\widetilde{\gamma}$ which joins u_- and u_+ and it fulfills (5.5) meaning that $u_0 \in W^{1,p}(\Omega)$ obtained from the mountain-pass theorem is nontrivial. That means that we have found a sign-changing weak solution u_0 of our original problem (1.1) which lies between u_- and u_+. That finishes the proof of the theorem. □

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Resonant $(p, 2)$-equations with concave terms

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We consider a nonlinear, nonhomogeneous parametric elliptic Dirichlet equation driven by the sum of a p-Laplacian and a Laplacian (so-called $(p,2)$-equation) and with a nonlinearity involving a concave term which enters with a negative sign. By applying variational methods along with truncation and comparison techniques as well as Morse theory (critical groups), we show that the problem under consideration has at least five nontrivial solutions (four of them have constant sign) for all sufficiently small values of the parameter.

Keywords: concave term; resonance; nonlinear regularity; nonlinear maximum principle; critical groups

AMS Subject Classifications: 35J20; 35J60; 35J92; 58E05

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $1 < q < 2 < p < \infty$. We study the following nonlinear nonhomogeneous parametric Dirichlet problem

$$\left.\begin{aligned} -\Delta_p u - \Delta u &= f(x,u) - \lambda|u|^{q-2}u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{P}_\lambda$$

where Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(\|\nabla u\|_{\mathbb{R}^N}^{p-2}\nabla u\right) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Here $\lambda > 0$ is a parameter to be specified and since $1 < q < 2 < p < \infty$, the right-hand side in problem $(P)_\lambda$ contains a concave term given through $-\lambda|u|^{q-2}u$. The perturbation $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is, $x \mapsto f(x,s)$ is measurable for all $s \in \mathbb{R}$ and $s \mapsto f(x,s)$ is continuous for a.a. $x \in \Omega$) being $(p-1)$-linear near $\pm\infty$ and resonance can occur with respect to the principal eigenvalue $\hat{\lambda}_1(p) > 0$ of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$. The aim of this work is to prove the existence of multiple solutions as the parameter $\lambda > 0$ varies.

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The study of elliptic problems with concave nonlinearities started with the seminal work of Ambrosetti et al. [1], who examined a semilinear equation driven by the Dirichlet Laplacian and with a parametric reaction of the special form

$$f(s) = \lambda |s|^{q-2}s + |s|^{r-2}s, \tag{1.1}$$

where

$$1 < q < 2 < r < 2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geq 3 \\ +\infty & \text{if } N = 1, 2 \end{cases}.$$

In (1.1), we have the competing effects of two distinct nonlinearities of different nature meaning that there is a concave term $\lambda |s|^{q-2}s$ and also a convex one $|s|^{r-2}s$. The authors of [1] were interested to find positive solutions and proved that the problem has two positive solutions provided $\lambda > 0$ is sufficiently small. Additional results for problems with combined nonlinearities as above were obtained by Bartsch and Willem [2], Li et al. [3], and Wang [4]. Extensions to equations driven by the Dirichlet p-Laplacian can be found in García Azorero et al. [5], Gasiński and Papageorgiou [6], Guo and Zhang [7], Hu and Papageorgiou [8], and Marano and Papageorgiou [9]. In all of the aforementioned works, the parametric concave term enters in the reaction with a positive sign. In our problem $(P)_\lambda$, the parametric concave term appears in the reaction with a negative sign. This produces a different geometry for the problem and therefore leads to a different multiplicity theorem. Semilinear problems with such a concave term were investigated by de Paiva and Massa [10], Papageorgiou and Rădulescu [11], and Perera [12].

We mention that equations involving the sum of a p-Laplacian and a Laplacian (also known as $(p, 2)$-equations) arise in mathematical physics; see, for example, the works of Benci et al. [13] (quantum physics) and Cherfils and Il′yasov [14] (plasma physics).

Our approach is variational based on critical point theory coupled with suitable truncation and comparison techniques as well as Morse theory (critical groups). In the next section, for the reader's convenience, we review the main mathematical tools that we will use in the sequel.

2. Preliminaries

Let X be a Banach space and X^* its topological dual while $\langle \cdot, \cdot \rangle$ denotes the duality brackets to the pair (X^*, X).

Definition 2.1 The functional $\varphi \in C^1(X)$ fulfills the Palais-Smale condition (the PS-condition for short) if the following holds: Every sequence $(u_n)_{n\geq 1} \subseteq X$ such that $(\varphi(u_n))_{n\geq 1}$ is bounded in $\mathbb{R}$ and $\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

This compactness-type condition on φ leads to a deformation theorem which is the main ingredient in the minimax theory of the critical values of φ. A basic result in that theory is the so-called mountain pass theorem.

THEOREM 2.2 *Let $\varphi \in C^1(X)$ be a functional satisfying the PS-condition and let $u_1, u_2 \in X$, $\|u_2 - u_1\|_X > \rho > 0$,*

$$\max\{\varphi(u_1), \varphi(u_2)\} < \inf\{\varphi(u) : \|u - u_1\|_X = \rho\} =: m_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_1, \gamma(1) = u_2\}$. Then $c \ge m_\rho$ with c being a critical value of φ.

In the analysis of problem $(P)_\lambda$ in addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will also use the ordered Banach space

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\right\}$$

and its positive cone

$$C_0^1(\overline{\Omega})_+ = \left\{u \in C_0^1(\overline{\Omega}) : u(x) \ge 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) = \left\{u \in C_0^1(\overline{\Omega}) : u(x) > 0\ \forall x \in \Omega,\ \text{and } \frac{\partial u}{\partial n}(x) < 0\ \forall x \in \partial\Omega\right\},$$

where $n = n(x)$ is the outer unit normal at $x \in \partial\Omega$.

Throughout this paper we denote the norm of $W_0^{1,p}(\Omega)$ by $\|\cdot\|_{W_0^{1,p}(\Omega)}$ and thanks to the Poincaré inequality it holds $\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_p$ for all $u \in W_0^{1,p}(\Omega)$, where $\|\cdot\|_p$ stands for the usual L^p-norm. The norm of $\mathbb{R}^N$ is denoted by $\|\cdot\|_{\mathbb{R}^N}$ and $(\cdot,\cdot)_{\mathbb{R}^N}$ stands for the inner product of $\mathbb{R}^N$.

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying a subcritical growth with respect to the second argument, that is

$$|f_0(x, s)| \le a(x)\left(1 + |s|^{r-1}\right) \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+$, and $1 < r < p^*$, where p^* is the critical exponent of p given by

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \ge N. \end{cases}$$

Let $F_0(x, s) = \int_0^s f_0(x, t)dt$ and let $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\varphi_0(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F_0(x, u)dx.$$

The next result is a special case of a more general theorem of Aizicovici et al. [15] and essentially is an outgrowth of the nonlinear regularity theory (see Ladyzhenskaya and Ural′tseva [16], Lieberman [17]).

THEOREM 2.3 *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, i.e. there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \quad \text{for all } h \in C_0^1(\overline{\Omega}) \text{ with } \|h\|_{C_0^1(\overline{\Omega})} \le \rho_0,$$

then $u_0 \in C_0^{1,\beta}(\overline{\Omega})$ *for some* $\beta \in (0,1)$ *and* u_0 *is also a local* $W_0^{1,p}(\Omega)$*-minimizer of* φ_0*, i.e. there exists* $\rho_1 > 0$ *such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ with } \|h\|_{W_0^{1,p}(\Omega)} \leq \rho_1.$$

Remark 1 The first result in this direction was obtained by Brezis and Nirenberg [18]. Subsequently, important extensions were proved by García Azorero et al. [5], Guo and Zhang [7], and Winkert [19].

Given $1 < r < \infty$, we denote by $\Delta_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ with $\frac{1}{r} + \frac{1}{r'} = 1$ the r-Laplacian defined by

$$\langle \Delta_r u, v \rangle = \int_\Omega \|\nabla u\|_{\mathbb{R}^N}^{r-2} (\nabla u, \nabla v)_{\mathbb{R}^N} dx \quad \text{for all } u, v \in W_0^{1,r}(\Omega). \tag{2.1}$$

If $r = 2$, then $\Delta_r = \Delta$ becomes the well-known Laplace operator and we have $\Delta \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$, where $\mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$ denotes the vector space of all bounded linear operators from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. The next proposition summarizes the main properties of the map $-\Delta_r$ (see Gasiński and Papageorgiou [20]).

Proposition 2.4 *If* $\Delta_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ *with* $1 < r < \infty$, $\frac{1}{r} + \frac{1}{r'} = 1$, *is defined by* (2.1)*, then* Δ_r *is bounded (in the sense that it maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone) and of type* $(S)_+$*, i.e. if* $u_n \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$ *and* $\limsup_{n\to\infty} \langle -\Delta_r u_n, u_n - u \rangle \leq 0$*, then* $u_n \to u$ *in* $W_0^{1,p}(\Omega)$.

Let $\hat{\lambda}_1(p)$ be the first eigenvalue of the negative Dirichlet p-Laplacian $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$ which has the subsequent properties:

- $\hat{\lambda}_1(p)$ is positive, simple, and isolated;
- $$\hat{\lambda}_1(p) = \inf\left[\frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0\right]. \tag{2.2}$$

The infimum in (2.2) is realized on the one-dimensional eigenspace whose elements do not change sign which easily follows from the representation in (2.2). Denote by $\hat{u}_1(p)$ the L^p-normalized eigenfunction (i.e. $\|\hat{u}_1(p)\|_p = 1$) associated to $\hat{\lambda}_1(p)$, the nonlinear regularity theory implies that $\hat{u}_1(p) \in C_0^1(\overline{\Omega})$ and the usage of the nonlinear maximum principle (see Gasiński and Papageorgiou [20, p.737–738]) yields $\hat{u}_1(p) \in \text{int}\left(C_0^1(\overline{\Omega})_+\right)$.

In addition to $\hat{\lambda}_1(p) > 0$, the Lusternik–Schnirelmann minimax scheme gives a whole strictly increasing sequence $\left(\hat{\lambda}_k(p)\right)_{k\geq 1}$ of eigenvalues of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$ such that $\hat{\lambda}_k(p) \to +\infty$ as $k \to \infty$. If $p \neq 2$, we do not know if this sequence exhausts the whole spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$ but in case $N = 1$ (ordinary differential equations) or $p = 2$ (linear eigenvalue problem), the answer is positive. In the case $p = 2$, we denote by $E\left(\hat{\lambda}_k(2)\right), k \geq 1$, the finite-dimensional eigenspace corresponding to the eigenvalue $\hat{\lambda}_k(2)$. Applying classical regularity theory, we have that $E\left(\hat{\lambda}_k(2)\right) \subseteq C_0^1(\overline{\Omega})$ for all $k \geq 1$

and the eigenspace has the so-called unique continuation property (ucp for short) meaning that if $u \in E\left(\hat{\lambda}_k(2)\right)$ vanishes on a set of positive Lebesgue measure, then $u(x) = 0$ for all $x \in \overline{\Omega}$. For every $k \geq 1$ we set

$$\overline{H}_k = \bigoplus_{i=1}^{k} E\left(\hat{\lambda}_i(2)\right) \quad \text{and} \quad \hat{H}_k = \overline{H}_k^{\perp} = \overline{\bigoplus_{i \geq k+1} E\left(\hat{\lambda}_i(2)\right)}.$$

In the linear case, we have a variational characterization for all eigenvalues, namely

$$\hat{\lambda}_1(2) = \inf\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega), u \neq 0\right] \tag{2.3}$$

and for $k \geq 2$

$$\begin{aligned}\hat{\lambda}_k(2) &= \max\left[\frac{\|\nabla \overline{u}\|_2^2}{\|\overline{u}\|_2^2} : \overline{u} \in \overline{H}_k, \overline{u} \neq 0\right] \\ &= \min\left[\frac{\|\nabla \hat{u}\|_2^2}{\|\hat{u}\|_2^2} : \hat{u} \in \hat{H}_{k-1}, \hat{u} \neq 0\right].\end{aligned} \tag{2.4}$$

Taking into account the ucp of the eigenspaces along with (2.3), (2.4), we obtain the subsequent lemma.

LEMMA 2.5

(a) *If $k \geq 1$, $\vartheta \in L^\infty(\Omega)_+$, $\vartheta(x) \leq \hat{\lambda}_k(2)$ a.e. in Ω with $\vartheta \neq \hat{\lambda}_k(2)$, then there exists $\hat{\xi}_0 > 0$ such that*

$$\left\|\nabla \hat{u}\right\|_2^2 - \int_\Omega \vartheta \hat{u}^2 dx \geq \hat{\xi}_0 \left\|\hat{u}\right\|_{H_0^1(\Omega)}^2 \quad \text{for all } \hat{u} \in \hat{H}_{k-1}.$$

(b) *If $k \geq 1$, $\vartheta \in L^\infty(\Omega)_+$, $\vartheta(x) \geq \hat{\lambda}_k(2)$ a.e. in Ω with $\vartheta \neq \hat{\lambda}_k(2)$, then there exists $\hat{\xi}_1 > 0$ such that*

$$\|\nabla \overline{u}\|_2^2 - \int_\Omega \vartheta \overline{u}^2 dx \leq -\hat{\xi}_1 \|\overline{u}\|_{H_0^1(\Omega)}^2 \quad \text{for all } \overline{u} \in \overline{H}_k.$$

Using the properties of $\hat{\lambda}_1(p)$, we derive the following result (see, e.g. Papageorgiou and Kyritsi [21, p.356]).

LEMMA 2.6 *Let $\vartheta \in L^\infty(\Omega)_+$ be such that $\vartheta(x) \leq \hat{\lambda}_1(p)$ a.e. in Ω and $\vartheta \neq \hat{\lambda}_1(p)$. Then there exists a number $\hat{\xi}_2 > 0$ such that*

$$\|\nabla u\|_p^p - \int_\Omega \vartheta |u|^p dx \geq \hat{\xi}_2 \|u\|_{W_0^{1,p}(\Omega)}^p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Next, we briefly recall some basic definitions and facts about Morse theory related to critical points. To this end, let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets.

$$\begin{aligned} \varphi^c &= \{u \in X : \varphi(u) \leq c\} && \text{(the sublevel set of } \varphi \text{ at } c), \\ K_\varphi &= \{u \in X : \varphi'(u) = 0\} && \text{(the critical set of } \varphi), \\ K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\} && \text{(the critical set of } \varphi \text{ at the level } c). \end{aligned}$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, we denote by $H_k(Y_1, Y_2)$ the $k \overset{th}{=}$-relative singular homology group of the pair (Y_1, Y_2) with integer coefficients. The critical groups of φ at an isolated $u_0 \in K_\varphi^c$ are defined by

$$C_k(\varphi, u_0) = H_k\left(\varphi^c \cap U, \varphi^c \cap U \setminus \{u_0\}\right) \quad \text{for all integers } k \geq 0,$$

where U is a neighborhood of u_0 such that $K_\varphi \cap \varphi^c \cap U = \{u_0\}$. The excision property of singular homology theory implies that the definition of critical groups above is independent of the particular choice of the neighborhood U.

If $u_0 \in X$ is a local minimizer of φ, then

$$C_k(\varphi, u_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0, \tag{2.5}$$

where $\delta_{k,0}$ is the Kronecker symbol, that is

$$\delta_{k,0} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases}.$$

For $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. It is well known that

$$u^\pm \in W_0^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

The Lebesgue measure on $\mathbb{R}^N$ will be denoted by $|\cdot|_N$. Finally, for any Carathéodory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ we define the Nemytskij operator $N_h : L^p(\Omega) \to (L^p(\Omega))^*$ corresponding to the function h by

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)).$$

3. Constant sign solutions

In this section, we prove the existence of constant sign solutions for problem $(P)_\lambda$. We impose the following conditions on the perturbation $f : \Omega \times \mathbb{R} \to \mathbb{R}$.

H$_1$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x, 0) = 0$ for a.a. $x \in \Omega$ and

(i) $|f(x, s)| \leq a(x)\left(1 + |s|^{p-1}\right)$ for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, and with $a \in L^\infty(\Omega)_+$;

(ii) $\limsup_{s\to\pm\infty} \frac{f(x,s)}{|s|^{p-2}s} \leq \hat{\lambda}_1(p)$ uniformly for a.a. $x \in \Omega$ and there exists $\xi_0 > 0$ such that

$$f(x,s)s - pF(x,s) \geq -\xi_0 \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R},$$

where $F(x,s) = \int_0^s f(x,t)\mathrm{d}t$;

(iii) there exist functions $\eta, \hat{\eta} \in L^\infty(\Omega)_+$ such that

$$\hat{\lambda}_1(2) \leq \eta(x) \quad \text{a.e. in } \Omega,\ \eta \neq \hat{\lambda}_1(2)$$

and

$$\eta(x) \leq \liminf_{s\to 0} \frac{f(x,s)}{s} \leq \limsup_{s\to 0} \frac{f(x,s)}{s} \leq \hat{\eta}(x)$$

uniformly for a.a. $x \in \Omega$;

(iv) $f(x,\cdot)$ is locally lower Lipschitz for a.a. $x \in \Omega$, that is, for every compact set $K \subseteq \mathbb{R}$, there exists a constant $c_K > 0$ such that

$$f(x,s_1) - f(x,s_2) \geq -c_K|s_1 - s_2| \quad \text{for all } s_1, s_2 \in K.$$

Remark 1 Hypothesis H_1(ii) implies that we can have resonance asymptotically at $\pm\infty$ with respect to $\hat{\lambda}_1(p) > 0$.

In order to prove the existence of constant sign solutions, we consider the positive and negative truncations of the reaction in problem $(P)_\lambda$ for $\lambda > 0$, namely the Carathéodory functions

$$g_\lambda^\pm(x,s) = f(x,\pm s^\pm) \mp \lambda \left(s^\pm\right)^{q-1}.$$

We set $G_\lambda^\pm(x,s) = \int_0^s g_\lambda^\pm(x,t)dt$ and consider the C^1-functionals $\varphi_\lambda^\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda^\pm(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega G_\lambda^\pm(x,u)dx.$$

The corresponding energy functional $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ to problem $(P)_\lambda$ is defined by

$$\varphi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda}{q}\|u\|_q^q - \int_\Omega F(x,u)dx,$$

which is of class C^1 as well. First, we will see that the functionals stated above are coercive.

PROPOSITION 3.1 *Let hypotheses H_1 be satisfied and let $\lambda > 0$. Then the functionals $\varphi_\lambda^\pm$ and φ_λ are coercive.*

Proof We will show the proof only for φ_λ^+, the proofs for the other functionals work similarly. Arguing by contradiction we suppose that φ_λ^+ is not coercive. Then we find a sequence $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ and a number $M_1 > 0$ such that

$$\|u_n\|_{W_0^{1,p}(\Omega)} \to \infty \quad \text{and} \quad \varphi_\lambda^+(u_n) \leq M_1.$$

The second relation gives

$$\frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{2}\|\nabla u_n\|_2^2 - \int_\Omega G_\lambda^+(x,u_n)dx \leq M_1 \quad \text{for all } n \geq 1. \tag{3.1}$$

Taking $y_n = \frac{u_n}{\|u_n\|_{W_0^{1,p}(\Omega)}}$ implies $\|y_n\|_{W_0^{1,p}(\Omega)} = 1$ and we may assume that

$$y_n \rightharpoonup y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^p(\Omega) \tag{3.2}$$

with some $y \in W_0^{1,p}(\Omega)$. Applying the representation of y_n inequality (3.1) becomes

$$\frac{1}{p}\|\nabla y_n\|_p^p - \int_\Omega \frac{G_\lambda^+(x,u_n)}{\|u_n\|_{W_0^{1,p}(\Omega)}^p}dx \leq \frac{M_1}{\|u_n\|_{W_0^{1,p}(\Omega)}^p} \quad \text{for all } n \geq 1. \tag{3.3}$$

Because of hypothesis H_1(i) we have that

$$\left(\frac{G_\lambda^+(\cdot,u_n(\cdot))}{\|u_n\|_{W_0^{1,p}(\Omega)}^p}\right)_{n\geq 1} \subseteq L^1(\Omega) \text{ is uniformly integrable.}$$

Taking into account the Dunford-Pettis theorem along with assumption H_1(ii) we obtain

$$\frac{G_\lambda^+(\cdot,u_n(\cdot))}{\|u_n\|_{W_0^{1,p}(\Omega)}^p} \rightharpoonup \frac{1}{p}\vartheta\left(y^+\right)^p \quad \text{in } L^2(\Omega) \tag{3.4}$$

with $\vartheta \in L^\infty(\Omega)$ satisfying $\vartheta(x) \leq \hat{\lambda}_1(p)$ a.e. in Ω. Passing to the limit in (3.3) as $n \to \infty$ and applying (3.2) as well as (3.4) yields

$$\|\nabla y\|_p^p \leq \int_\Omega \vartheta\left(y^+\right)^p dx, \tag{3.5}$$

which implies

$$\left\|\nabla y^+\right\|_p^p \leq \int_\Omega \vartheta\left(y^+\right)^p dx. \tag{3.6}$$

Suppose now that $\vartheta \neq \hat{\lambda}_1(p)$. Then from (3.6) and Lemma 2.6 we get $y^+ = 0$. So inequality (3.5) implies $y^- = 0$, that is $y = 0$. Then, using (3.3), we see that

$$y_n \to 0 \quad \text{in } W_0^{1,p}(\Omega),$$

a contradiction to the fact that $\|y_n\|_{W_0^{1,p}(\Omega)} = 1$ for all $n \geq 1$.

Now we assume that $\vartheta(x) = \hat{\lambda}_1(p)$ a.e. in Ω. Then (3.6) and (2.2) give

$$\left\|\nabla y^+\right\|_p^p = \hat{\lambda}_1(p)\left\|y^+\right\|_p^p$$

which means that

$$y^+ = \xi\hat{u}_1(p) \quad \text{for some } \xi \geq 0.$$

If $\xi = 0$, then $y^+ = 0$ and due to (3.5) $y = 0$. Hence, because of (3.3), $y_n \to 0$ in $W_0^{1,p}(\Omega)$ which is a contradiction since $\|y_n\|_{W_0^{1,p}(\Omega)} = 1$ for all $n \geq 1$.

If $\xi > 0$, then $y^+ \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and so $y^+(x) > 0$ for all $x \in \Omega$. Since y^+ is the limit of y_n^+ in $W_0^{1,p}(\Omega)$ (see (3.2)) and $y_n^+ = \frac{u_n^+}{\|u_n\|_{W_0^{1,p}(\Omega)}}$ it follows that

$$u_n^+(x) \to +\infty \quad \text{for a.a. } x \in \Omega. \tag{3.7}$$

Thanks to hypothesis H_1(ii) and since $q < p$ we further obtain for a.a. $x \in \Omega$ and for all $u > 0$

$$\begin{aligned}\frac{d}{du}\frac{G_\lambda^+(x,u)}{u^p} &= \frac{f(x,u)u^p - \lambda u^{q+p-1} - pF(x,u)u^{p-1} + \frac{\lambda p}{q}u^{q+p-1}}{u^{2p}} \\ &= \frac{f(x,u)u - \lambda u^q - pF(x,u) + \frac{\lambda p}{q}u^q}{u^{p+1}} \\ &\geq -\frac{\xi_0}{u^{p+1}}.\end{aligned}$$

We conclude

$$\frac{G_\lambda^+(x,y)}{y^p} - \frac{G_\lambda^+(x,u)}{u^p} \geq \frac{\xi_0}{p}\left[\frac{1}{y^p} - \frac{1}{u^p}\right] \tag{3.8}$$

for a.a. $x \in \Omega$ and for all $y \geq u > 0$. From hypothesis H_1(ii) we see at once that

$$\limsup_{s\to\pm\infty} \frac{pF(x,s)}{|s|^p} \leq \hat{\lambda}_1(p) \quad \text{uniformly for a.a. } x \in \Omega.$$

Then, passing in (3.8) to the limit as $y \to +\infty$, since $q < p$, we derive

$$\frac{\hat{\lambda}_1(p)}{p} - \frac{G_\lambda^+(x,u)}{u^p} \geq -\frac{\xi_0}{p}\frac{1}{u^p} \quad \text{for a.a. } x \in \Omega \text{ and for all } u > 0,$$

which implies

$$pF(x,u) - \frac{\lambda p}{q}u^q - \hat{\lambda}_1(p)u^p \leq \xi_0 \quad \text{for a.a. } x \in \Omega \text{ and for all } u \geq 0. \tag{3.9}$$

Inequality (3.1) can be written as

$$\frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{2}\|\nabla u_n\|_2^2 \leq M_1 + \int_\Omega G_\lambda^+(x,u_n)dx \quad \text{for all } n \geq 1,$$

which, due to (2.2) and (3.9), gives

$$\begin{aligned}\frac{p}{2}\hat{\lambda}_1(2)\left\|u_n^+\right\|_2^2 &\leq M_1 p + \int_\Omega \left[pF\left(x,u_n^+\right) - \frac{\lambda p}{q}\left(u_n^+\right)^q - \hat{\lambda}_1(p)\left(u_n^+\right)^p\right]dx \\ &\leq M_2\end{aligned}$$

with $M_2 = M_1 p + \xi_0|\Omega|_N > 0$ and for all $n \geq 1$. This implies

$$\int_\Omega \left(u_n^+\right)^2 dx \leq \frac{2M_2}{p\hat{\lambda}_1(2)} \quad \text{for all } n \geq 1. \tag{3.10}$$

On the other side, from (3.7) and Fatou's Lemma, we have

$$\int_\Omega \left(u_n^+\right)^2 dx \to +\infty \quad \text{as } n \to \infty. \tag{3.11}$$

Comparing (3.10) and (3.11), we reach a contradiction. This proves that φ_λ^+ is coercive. □

In general, coercivity does not imply the PS-condition (see, e.g. Gasiński and Papageorgiou [20, Example 5.1.15]). However, for the functionals $\varphi_\lambda^\pm$ and φ_λ, this implication is true as stated in the next proposition, which is a consequence of Proposition 2.2 of Marano and Papageorgiou [22]. For completeness we provide the proof.

PROPOSITION 3.2 *If $\varphi_\lambda^\pm$ and φ_λ are coercive, then they satisfy the PS-condition.*

Proof The proof will be given only for φ_λ^+, the other ones work similarly. Suppose $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is a PS-sequence, that is

$$\left|\varphi_\lambda^+(u_n)\right| \leq M_4 \quad \text{for some } M_4 > 0, \text{ for all } n \geq 1, \tag{3.12}$$

$$\left(\varphi_\lambda^+\right)'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{3.13}$$

The assertion in (3.12) along with the coercivity of φ_λ^+ implies that $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. Therefore, we may assume that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\Omega). \tag{3.14}$$

From (3.13) it follows

$$\left|\left\langle -\Delta_p u_n, h\right\rangle + \langle -\Delta u_n, h\rangle - \int_\Omega g_\lambda^+(x, u_n)h\,dx\right| \leq \varepsilon_n \|h\|_{W_0^{1,p}(\Omega)},$$

for all $h \in W_0^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$. Now, choosing $h = u_n - u \in W_0^{1,p}(\Omega)$, passing to the limit as $n \to \infty$, and using the convergence properties in (3.14), we obtain

$$\lim_{n\to\infty} \left[\left\langle -\Delta_p u_n, u_n - u\right\rangle + \langle -\Delta u_n, u_n - u\rangle\right] = 0,$$

which by the monotonicity of $-\Delta$ implies that

$$\limsup_{n\to\infty} \left[\left\langle -\Delta_p u_n, u_n - u\right\rangle + \langle -\Delta u, u_n - u\rangle\right] \leq 0.$$

Applying again (3.14) we infer from the last relation

$$\limsup_{n\to\infty} \left\langle -\Delta_p u_n, u_n - u\right\rangle \leq 0,$$

which by the $(S)_+$-property of $-\Delta_p$ (see Proposition 2.4) results in $u_n \to u$ in $W_0^{1,p}(\Omega)$. Hence, φ_λ^+ fulfills the PS-condition. □

PROPOSITION 3.3 *If hypotheses H_1 hold, then we can find $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ there exists $t^* = t^*(\lambda)$ for which*

$$\varphi_\lambda\left(\pm t^* \hat{u}_1(2)\right) < 0.$$

Proof Given $\varepsilon > 0$, by virtue of hypotheses H_1(i), (iii), there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$F(x, s) \geq \frac{1}{2}\left(\eta(x) - \varepsilon\right)s^2 - c_1|s|^p \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R}. \tag{3.15}$$

By means of (3.15) we have for $t > 0$

$$\begin{aligned}\varphi_\lambda\left(t\hat{u}_1(2)\right) &= \frac{t^p}{p}\left\|\nabla\hat{u}_1(2)\right\|_p^p + \frac{t^2}{2}\left\|\nabla\hat{u}_1(2)\right\|_2^2 + \frac{\lambda t^q}{q}\left\|\hat{u}_1(2)\right\|_q^q \\ &\quad - \int_\Omega F\left(x, t\hat{u}_1(2)\right)dx \\ &\leq \frac{t^2}{2}\left[\int_\Omega\left(\hat{\lambda}_1(2) - \eta(x)\right)\left(\hat{u}_1(2)\right)^2 dx + \varepsilon\right] + c_2\left[t^p + \lambda t^q\right]\end{aligned}$$

for some $c_2 > 0$. Thanks to hypothesis H_1(iii) and since $\hat{u}_1(2) \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ we conclude

$$\xi_* = \int_\Omega\left(\eta(x) - \hat{\lambda}_1(2)\right)\left(\hat{u}_1(2)\right)^2 dx > 0.$$

Choosing $\varepsilon \in (0, \xi_*)$ we have

$$\begin{aligned}\varphi_\lambda\left(t\hat{u}_1(2)\right) &\leq -c_3t^2 + c_2\left[t^p + \lambda t^q\right] \\ &= \left[c_2\left(t^{p-2} + \lambda t^{q-2}\right) - c_3\right]t^2 \qquad (3.16)\end{aligned}$$

for some $c_3 > 0$ and for all $t > 0$.

Let $\beta_\lambda(t) = t^{p-2} + \lambda t^{q-2}$ for all $t > 0$. Obviously, $\beta_\lambda \in C^1(0,\infty)$ and since $q < 2 < p$ it follows

$$\beta_\lambda(t) \to +\infty \quad \text{as } t \to 0^+ \text{ and as } t \to +\infty.$$

Hence, we find a number $t_0 \in (0, +\infty)$ such that

$$\beta\left(t_0\right) = \inf\left[\beta_\lambda(t) : t > 0\right] > 0.$$

Moreover, it holds

$$\beta'_\lambda(t_0) = \left[(p-2)t_0^{p-3} + \lambda(q-2)t_0^{q-3}\right] = 0,$$

which implies

$$t_0 = t_0(\lambda) = \left[\frac{\lambda(2-q)}{p-2}\right]^{\frac{1}{p-q}}.$$

We see that $\beta_\lambda(t_0(\lambda)) \to 0$ as $\lambda \to 0^+$. Therefore, there exists a number $\lambda^* > 0$ such that

$$\beta_\lambda\left(t_0\right) < \frac{c_3}{c_2} \quad \text{for all } \lambda \in \left(0, \lambda^*\right).$$

Taking $t^* = t^*(\lambda) = t_0(\lambda)$, inequality (3.16) gives

$$\varphi_\lambda\left(\pm t^*\hat{u}_1(2)\right) < 0.$$

□

The next proposition will be helpful in verifying the mountain pass geometry of the functionals $\varphi_\lambda^\pm$ and φ_λ.

PROPOSITION 3.4 *Let hypotheses H_1 be satisfied and let $\lambda > 0$. Then $u = 0$ is a local minimizer of the functionals $\varphi_\lambda^\pm$ and φ_λ.*

Proof As before we will do the proof only for φ_λ^+. By virtue of hypothesis H_1(iii) there exist numbers $c_4 > 0$ and $\delta > 0$ such that

$$F(x,s) \leq c_4 s^2 \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in [0,\delta]. \tag{3.17}$$

Let $u \in C_0^1(\overline{\Omega})$ satisfy $\|u\|_{C_0^1(\overline{\Omega})} \leq \delta$. Applying (3.17) it follows

$$\begin{aligned}\varphi_\lambda^+(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega G_\lambda^+(x,u)dx \\ &\geq \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \left[\frac{\lambda}{q} - c_4\|u\|_{C(\overline{\Omega})}^{2-q}\right]\left\|u^+\right\|_q^q \\ &\geq \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \left[\frac{\lambda}{q} - c_4\delta^{2-q}\right]\left\|u^+\right\|_q^q. \end{aligned} \tag{3.18}$$

Choosing $\delta > 0$ such that $\delta < \left(\frac{\lambda}{qc_4}\right)^{\frac{1}{2-q}}$ we infer from (3.18)

$$\varphi_\lambda^+(u) \geq 0 = \varphi_\lambda^+(0) \quad \text{for all } u \in C_0^1(\overline{\Omega}) \text{ with } \|u\|_{C_0^1(\overline{\Omega})} \leq \delta.$$

This means that $u = 0$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_λ^+ and because of Theorem 2.3 $u = 0$ is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_λ^+. The proofs for φ_λ^- and φ_λ can be done in the same way. □

Now, we will apply the mountain pass theorem (Theorem 2.2) and the direct method to prove the existence of at least four nontrivial constant sign solutions of $(P)_\lambda$ for all $\lambda > 0$ sufficiently small whereby two of the solutions have positive sign and the other ones have negative sign. In what follows $\lambda^* > 0$ denotes the number obtained in Proposition 3.3.

PROPOSITION 3.5 *Let hypotheses H_1 be satisfied and let $\lambda \in (0, \lambda^*)$. Then problem $(P)_\lambda$ admits at least four nontrivial solutions of constant sign, namely*

$$u_0, \hat{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\} \quad \text{and} \quad v_0, \hat{v} \in -\left(C_0^1(\overline{\Omega})_+\right) \setminus \{0\}$$

such that

$$u_0(x), \hat{u}(x) > 0 \quad \text{for all } x \in \Omega \quad \text{and} \quad v_0(x), \hat{v}(x) < 0 \quad \text{for all } x \in \Omega.$$

Moreover, u_0 and v_0 are the local minimizers of φ_λ.

Proof Taking into account Proposition 3.1, we know that φ_λ^+ is coercive for all $\lambda > 0$. Moreover, by applying the Sobolev embedding theorem we easily verify that φ_λ^+ is sequentially weakly lower semicontinuous as well. Therefore, by virtue of the Weierstrass theorem, there exists an element $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda^+(u_0) = \inf\left[\varphi_\lambda^+(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{3.19}$$

From Proposition 3.3 it follows that if $\lambda \in (0, \lambda^*)$ we can find a number $t^* = t^*(\lambda) > 0$ such that

$$\varphi_\lambda \left(t^* \hat{u}_1(2)\right) < 0,$$

which ensures, due to $\varphi_\lambda\big|_{C_0^1(\overline{\Omega})_+} = \varphi_\lambda^+\big|_{C_0^1(\overline{\Omega})_+}$ and $\hat{u}_1(2) \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, that

$$\varphi_\lambda^+ \left(t^* \hat{u}_1(2)\right) < 0.$$

Hence, because of (3.19), we obtain

$$\varphi_\lambda^+ (u_0) < 0 = \varphi_\lambda^+(0),$$

implying $u_0 \neq 0$. Since u_0 is a critical point of φ_λ^+ we have

$$\left(\varphi_\lambda^+\right)' (u_0) = 0,$$

that is,

$$\left\langle -\Delta_p u_0, h\right\rangle + \langle -\Delta u_0, h\rangle = \left\langle N_{g_\lambda^+}(u_0), h\right\rangle \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.20}$$

Taking $h = -u_0^- \in W_0^{1,p}(\Omega)$ as test function in (3.20) gives $u_0 \geq 0$. Therefore, (3.20) becomes

$$\left\langle -\Delta_p u_0, h\right\rangle + \langle -\Delta u_0, h\rangle = \left\langle N_f(u_0) - \lambda u_0^{q-1}, h\right\rangle \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

meaning that u_0 solves our original problem

$$\begin{array}{ll} -\Delta_p u_0 - \Delta u_0 = f(x, u_0) - \lambda u_0^{q-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{array} \tag{3.21}$$

Note that $u_0 \in L^\infty(\Omega)$ (see Ladyzhenskaya and Ural′tseva [16, p.286]) and by means of the regularity results of Lieberman [17, Theorem 1] we infer that $u_0 \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$.

Now, let $a : \mathbb{R}^N \to \mathbb{R}^N$ be the map defined by $a(\xi) = \|\xi\|_{\mathbb{R}^N}^{p-2}\xi + \xi$. Since $p > 2$ it is easy to see that $a \in C^1\left(\mathbb{R}^N, \mathbb{R}^N\right)$. There holds

$$\nabla a(\xi) = \|\xi\|_{\mathbb{R}^N}^{p-2}\left[I + (p-2)\frac{\xi \otimes \xi}{\|\xi\|_{\mathbb{R}^N}^2}\right] + I \quad \text{for all } \xi \in \mathbb{R}^N$$

and

$$(\nabla a(\xi) y, y)_{\mathbb{R}^N} \geq \|\xi\|_{\mathbb{R}^N}^2 \quad \text{for all } \xi, y \in \mathbb{R}^N.$$

Thanks to hypothesis H_1(iv) we may apply the tangency principle of Pucci and Serrin [23, p.35] which gives

$$u_0(x) > 0 \quad \text{for all } x \in \Omega.$$

Claim u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of φ_λ.

Arguing by contradiction, suppose we can find a sequence $(u_n)_{n\geq 1} \subseteq C_0^1(\overline{\Omega})$ such that

$$u_n \to u_0 \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{and} \quad \varphi_\lambda(u_n) < \varphi_\lambda(u_0).$$

Since $\varphi_\lambda\big|_{C_0^1(\overline{\Omega})_+} = \varphi_\lambda^+\big|_{C_0^1(\overline{\Omega})_+}$ and because of (3.19) it follows

$$\begin{aligned}
0 &> \varphi_\lambda(u_n) - \varphi_\lambda(u_0) \\
&= \varphi_\lambda(u_n) - \varphi_\lambda^+(u_0) \\
&\geq \varphi_\lambda(u_n) - \varphi_\lambda^+(u_n) \\
&= \frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{\lambda}{q}\|u_n\|_q^q - \int_\Omega F(x,u_n)dx \\
&\quad - \frac{1}{p}\|\nabla u_n\|_p^p - \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{\lambda}{q}\left\|u_n^+\right\|_q^q + \int_\Omega F\left(x,u_n^+\right)dx \\
&= \frac{\lambda}{q}\left\|u_n^-\right\|_q^q - \int_\Omega F\left(x,-u_n^-\right)dx. \qquad (3.22)
\end{aligned}$$

By virtue of hypotheses H_1(i)–(iii), there exist numbers $c_5, c_6 > 0$ such that

$$F(x,s) \leq c_5 s^2 + c_6|s|^p \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R}. \qquad (3.23)$$

Applying (3.23) in (3.22) yields

$$\begin{aligned}
0 &> \varphi_\lambda(u_n) - \varphi_\lambda(u_0) \\
&\geq \frac{\lambda}{q}\left\|u_n^-\right\|_q^q - c_5\left\|u_n^-\right\|_2^2 - c_6\left\|u_n^-\right\|_p^p \\
&\geq \frac{\lambda}{q}\left\|u_n^-\right\|_q^q - \left[c_5\left\|u_n^-\right\|_{C(\overline{\Omega})}^{2-q} + c_6\left\|u_n^-\right\|_{C(\overline{\Omega})}^{p-q}\right]\left\|u_n^-\right\|_q^q. \qquad (3.24)
\end{aligned}$$

Since $u_0 > 0$ we note that $u_n^- \to 0$ in $C(\overline{\Omega})$. Therefore, (3.24) implies the existence of a number $n_0 \geq 1$ such that

$$0 > \varphi_\lambda(u_n) - \varphi_\lambda(u_0) \geq 0 \quad \text{for all } n \geq n_0,$$

which is a contradiction. This proves the Claim.

Taking into account the Claim and Theorem 2.3, we obtain that u_0 is a $W_0^{1,p}(\Omega)$-minimizer of φ_λ^+.

From Proposition 3.4, we know that $u = 0$ is a local minimizer of φ_λ^+. We may assume that it is an isolated critical point of φ_λ^+ or otherwise we have a whole sequence of distinct positive solutions of $(P)_\lambda$. Then, from Aizicovici et al. [24, Proof of Proposition 29] (see also de Figueiredo [25, Theorem 5.10, p.42]) we can find a number $\rho \in \left(0, \|u_0\|_{W_0^{1,p}(\Omega)}\right)$ sufficiently small such that

$$\varphi_\lambda^+(u_0) < 0 = \varphi_\lambda^+(0) < \inf\left[\varphi_\lambda^+(u) : \|u\|_{W_0^{1,p}(\Omega)} = \rho\right] = m_\rho^+. \qquad (3.25)$$

Recall that φ_λ^+ is coercive (see Proposition 3.1). So, Proposition 3.2 implies that φ_λ^+ fulfills the PS-condition. This fact along with (3.25) permit the usage of the mountain pass theorem stated in Theorem 2.2 to obtain an element $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\varphi_\lambda^+} \quad \text{and} \quad m_\rho^+ \leq \varphi_\lambda^+\left(\hat{u}\right). \qquad (3.26)$$

Since $\hat{u} \in K_{\varphi_\lambda^+}$ we have $\left(\varphi_\lambda^+\right)'(\hat{u}) = 0$, that is

$$\left\langle -\Delta_p \hat{u}, h\right\rangle + \left\langle -\Delta \hat{u}, h\right\rangle = \left\langle N_{g_\lambda^+}(\hat{u}), h\right\rangle \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.27}$$

Taking $h = -\hat{u}^- \in W_0^{1,p}(\Omega)$ in (3.27) gives $\left\|\nabla \hat{u}^-\right\|_p^p + \left\|\nabla \hat{u}^-\right\|_2^2 = 0$. Thus, $\hat{u} \geq 0$. From (3.25) and (3.26), it follows that $\hat{u} \notin \{0, u_0\}$ and $\hat{u}$ is a positive solution of $(P)_\lambda$ with $\lambda \in (0, \lambda^*)$. As before the nonlinear regularity theory and the tangency principle imply that $\hat{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$ with $\hat{u}(x) > 0$ for all $x \in \Omega$.

Similarly, working with φ_λ^- instead of φ_λ^+, we show the existence of two negative constant sign solutions $v_0, \hat{v} \in -\left(C_0^1(\overline{\Omega})_+\right) \setminus \{0\}$ with $v_0(x), \hat{v}(x) < 0$ for all $x \in \Omega$. □

4. Five nontrivial solutions

In this section, we have to strengthen the hypotheses of the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ in order to prove the existence of a fifth nontrivial solution of problem $(P)_\lambda$ for all $\lambda > 0$ sufficiently small. We suppose the following conditions.

H_2: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that $f(x, 0) = 0$ for a.a. $x \in \Omega$, $f(x, \cdot) \in C^1(\mathbb{R})$ and

(i) $|f_s'(x, s)| \leq a(x)\left(1 + |s|^{p-2}\right)$ for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, and with $a \in L^\infty(\Omega)_+$;

(ii) $\limsup_{s \to \pm\infty} \frac{f(x,s)}{|s|^{p-2}s} \leq \hat{\lambda}_1(p)$ uniformly for a.a. $x \in \Omega$ and there exists $\xi_0 > 0$ such that

$$f(x, s)s - pF(x, s) \geq -\xi_0 \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R},$$

where $F(x, s) = \int_0^s f(x, t)dt$;

(iii) there exist an integer $m \geq 3$ such that

$$f_s'(x, 0) \in \left[\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)\right] \quad \text{a.e. in } \Omega,$$

with $f_s'(\cdot, 0) \neq \hat{\lambda}_m(2)$, $f_s'(\cdot, 0) \neq \hat{\lambda}_{m+1}(2)$ and

$$f_s'(x, 0) = \lim_{s \to 0} \frac{f(x, s)}{s} \quad \text{uniformly for a.a. } x \in \Omega;$$

(iv) $|F(x, s)| \leq \frac{\hat{\lambda}_{m+1}(2)}{2} s^2 + \frac{\hat{\lambda}_1(p)}{p}|s|^p$ for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

Remark 1 The differentiability of $f(x, \cdot)$ along with hypothesis H_2(i) imply that $f(x, \cdot)$ is locally Lipschitz.

Let

$$V_m = \bigoplus_{i=1}^m E\left(\hat{\lambda}_i(2)\right) \quad \text{and} \quad W_m = W_0^{1,p}(\Omega) \cap V_m^\perp.$$

Then we have

$$W_0^{1,p}(\Omega) = V_m \bigoplus W_m \quad \text{and} \quad d_m = \dim V_m < \infty.$$

In what follows let

$$\partial B_\rho^m = \left\{ u \in V_m : \|u\|_{W_0^{1,p}(\Omega)} = \rho \right\}, \quad \rho > 0.$$

PROPOSITION 4.1 *If hypotheses H_2 hold, then we can find $\lambda_0^* \in (0, \lambda^*]$, where $\lambda^* > 0$ is as in Proposition 3.3, such that for all $\lambda \in \left(0, \lambda_0^*\right)$ there exists a number $\rho = \rho(\lambda) > 0$ for which*

$$\sup\left[\varphi_\lambda(u) : u \in \partial B_\rho^m\right] < 0.$$

Proof Let $\eta(x) = f_s'(x, 0)$. Given $\varepsilon > 0$, by virtue of hypotheses H_2(i),(iii), there exists a number $c_7 = c_7(\varepsilon) > 0$ such that

$$F(x, s) \geq \frac{1}{2}\left(\eta(x) - \varepsilon\right) s^2 - c_7 |s|^p \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R}. \tag{4.1}$$

Taking into account (4.1), we obtain for $u \in V_m$

$$\begin{aligned}
\varphi_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda}{q}\|u\|_q^q - \int_\Omega F(x, u)dx \\
&\leq \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda}{q}\|u\|_q^q - \frac{1}{2}\int_\Omega \eta(x)u^2 dx \\
&\quad + \frac{\varepsilon}{2}\|u\|_2^2 + c_7\|u\|_p^p \\
&= \frac{1}{2}\left[\|\nabla u\|_2^2 - \int_\Omega \eta(x)u^2 dx + \varepsilon\|u\|_2^2\right] \\
&\quad + \left[\frac{1}{p}\|\nabla u\|_p^p + \frac{\lambda}{q}\|u\|_q^q + c_7\|u\|_p^p\right].
\end{aligned} \tag{4.2}$$

Because of $u \in V_m$ and due to hypothesis H_2(iii) along with Lemma 2.5(b) we verify that

$$\|\nabla u\|_2^2 - \int_\Omega \eta(x)u^2 dx \leq -\hat{\xi}_1 \|u\|_{H_0^1(\Omega)}^2.$$

Since V_m is finite dimensional, it is clear that all norms of V_m are equivalent. Therefore, from (4.2) and for $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned}
\varphi_\lambda(u) &\leq -c_8\|u\|_{W_0^{1,p}(\Omega)}^2 + c_9\left(\lambda\|u\|_{W_0^{1,p}(\Omega)}^q + \|u\|_{W_0^{1,p}(\Omega)}^p\right) \\
&= \left[-c_8 + c_9\left(\lambda\|u\|_{W_0^{1,p}(\Omega)}^{q-2} + \|u\|_{W_0^{1,p}(\Omega)}^{p-2}\right)\right]\|u\|_{W_0^{1,p}(\Omega)}^2
\end{aligned}$$

for some $c_8 = c_8(\varepsilon)$, $c_9 > 0$.

We consider the function $\hat{\beta}_\lambda(t) = \lambda t^{q-2} + t^{p-2}$ and recall that $q < 2 < p$. As in the proof of Proposition 3.3 we can find $\hat{\lambda}^* > 0$ such that for all $\lambda \in \left(0, \hat{\lambda}^*\right)$ there exists $\rho = \rho(\lambda) > 0$ for which

$$\varphi_\lambda(u) < 0 \quad \text{for all } u \in \partial B_\rho^m.$$

Taking $\lambda_0^* = \min\left\{\hat{\lambda}^*, \lambda^*\right\}$ proves the assertion of the proposition. □

We have another useful result.

PROPOSITION 4.2 *Let hypotheses H_2 be satisfied and let $\lambda > 0$. Then there holds $\varphi_\lambda\big|_{W_m} \geq 0$.*

Proof Taking into account (2.2), (2.4), as well as hypothesis H_2(iv) we have for $u \in W_m$

$$\begin{aligned}\varphi_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda}{q}\|u\|_q^q - \int_\Omega F(x,u)dx \\ &\geq \frac{1}{p}\left[\|\nabla u\|_p^p - \hat{\lambda}_1(p)\|u\|_p^p\right] + \frac{1}{2}\left[\|\nabla u\|_2^2 - \hat{\lambda}_{m+1}(2)\|u\|_2^2\right] \\ &\geq 0.\end{aligned}$$

□

Now we are ready to prove the complete multiplicity theorem concerning problem $(P)_\lambda$ for all $\lambda > 0$ sufficiently small.

THEOREM 4.3 *If hypotheses H_2 hold, then there exists $\lambda_0^* > 0$ such that for all $\lambda \in \left(0, \lambda_0^*\right]$ problem $(P)_\lambda$ admits at least five distinct nontrivial solutions*

$$u_0, \hat{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\}, \quad v_0, \hat{v} \in -\left(C_0^1(\overline{\Omega})_+\right) \setminus \{0\}, \quad \text{and} \quad y_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$$

such that

$$u_0(x), \hat{u}(x) > 0 \quad \text{for all } x \in \Omega \quad \text{and} \quad v_0(x), \hat{v}(x) < 0 \quad \text{for all } x \in \Omega.$$

Proof As it is always the case in multiplicity theorems, we assume that the energy functional φ_λ has a finite critical set or otherwise we already have a fifth solution and so we are done (recall that the critical points of the energy functional are solutions of our problem). From Proposition 3.5, we know that we can find $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem $(P)_\lambda$ has at least four nontrivial constant sign solutions

$$u_0, \hat{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\} \quad \text{and} \quad v_0, \hat{v} \in -\left(C_0^1(\overline{\Omega})_+\right) \setminus \{0\}$$

such that

$$u_0(x), \hat{u}(x) > 0 \quad \text{for all } x \in \Omega \quad \text{and} \quad v_0(x), \hat{v}(x) < 0 \quad \text{for all } x \in \Omega.$$

From Proposition 3.5, we know that u_0 and v_0 are local minimizers of φ_λ. Hence, due to (2.5),

$$C_k(\varphi_\lambda, u_0) = C_k(\varphi_\lambda, v_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.3}$$

Furthermore, the proof of Proposition 3.5 shows that $\hat{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$ and $\hat{v} \in -\left(C_0^1(\overline{\Omega})_+\right) \setminus \{0\}$ are critical points of φ_λ^+ and φ_λ^-, respectively, of mountain-pass type such that

$$0 < m_\rho^+ \leq \varphi_\lambda\left(\hat{u}\right) \quad \text{and} \quad 0 < m_\rho^- \leq \varphi_\lambda\left(\hat{v}\right). \tag{4.4}$$

Since φ_λ is coercive (see Proposition 3.1), it is bounded from below. This fact along with Propositions 4.1, 4.2 imply the existence of $\lambda_0^* \in (0, \lambda^*]$ such that for all $\lambda \in \left(0, \lambda_0^*\right)$ φ_λ fulfills the assumptions of Theorem 3.1 in Perera [12]. Hence, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_{\varphi_\lambda}, \quad \varphi_\lambda\left(y_0\right) < 0 = \varphi_\lambda(0), \quad \text{and} \quad C_{d_m-1}\left(\varphi_\lambda, y_0\right) \neq 0. \tag{4.5}$$

From (4.5), it follows that y_0 is a nontrivial solution of $(P)_\lambda$ for all $\lambda \in \left(0, \lambda_0^*\right)$. Since $m \geq 3$ we note that $d_m - 1 \geq 2$. Therefore, from (4.3) and (4.5), we conclude that $y_0 \notin \{u_0, v_0\}$ and from (4.4) and (4.5) it follows that $y_0 \notin \left\{\hat{u}, \hat{v}\right\}$. Finally, as before, the nonlinear regularity theory implies $y_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$. This finishes the proof. □

Remark 2 In contrast to the problems where the concavity enters in the nonlinearity with a positive sign (see Gasiński and Papageorgiou [6] and Hu and Papageorgiou [8]), here we are unable to show that the fifth solution y_0 is nodal. It is an interesting open problem whether y_0 has changing sign. Finally, we mention that we could have used the differential operator $-\Delta_p u - \mu \Delta u$ with $\mu > 0$ without any problem. For simplicity in the presentation, we have assumed that $\mu = 1$.

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On a Parametric Nonlinear Dirichlet Problem with Subdiffusive and Equidiffusive Reaction

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Abstract

We study a nonlinear parametric elliptic equation (nonlinear eigenvalue problem) driven by a nonhomogeneous differential operator. Our setting incorporates equations driven by the p-Laplacian, the (p,q)-Laplacian, and the generalized p-mean curvature differential operator. Applying variational methods we show that for $\lambda > 0$ (the parameter) sufficiently large the problem has at least three nontrivial smooth solutions whereby one is positive, one is negative and the last one has changing sign (nodal). In the particular case of $(p,2)$-equations, using Morse theory, we produce another nodal solution for a total of four nontrivial smooth solutions.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $1 < q \leq p$. We study the following nonlinear nonhomogeneous eigenvalue problem

$$\begin{aligned} -\operatorname{div} a(\nabla u) &= \lambda |u|^{q-2}u - f(x,u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{P}_\lambda$$

where the mapping $a : \mathbb{R}^N \to \mathbb{R}^N$ is supposed to be continuous and strictly monotone with $(p-1)$-growth. The precise regularity conditions are presented in hypotheses H(a) (see Section 2). These conditions incorporate in our framework of analysis some important differential operators, such as the p-Laplace differential operator ($1 < p < \infty$), the (p,q)-differential operator ($1 < q < p$), and the generalized p-mean curvature differential operator ($1 < p < \infty$). Furthermore, $\lambda > 0$ is a parameter to be specified later and the perturbation $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., $x \mapsto f(x,s)$ is measurable for all $s \in \mathbb{R}$ and $s \mapsto f(x,s)$ is continuous for a.a. $x \in \Omega$) whereby it is assumed that $f(x,\cdot)$ exhibits a $(p-1)$-superlinear growth near $\pm\infty$ for a.a. $x \in \Omega$. We note that in case $f(x,s) = f(s) = |s|^{r-2}s$, $q \leq p < r < \infty$, and $-\operatorname{div} a(\nabla(\cdot)) = -\Delta_p$ (the negative p-Laplacian), problem $(P)_\lambda$ becomes the so-called p-logistic equation which is important in mathematical biology and describes the dynamics of biological populations whose mobility is density-dependent. If $q < p$, then we have the subdiffusive case while the equidiffusive case occurs for $q = p$. Since the solution of these applications describes population density, positive solutions are of concern (see Gurtin-MacCamy [24]). Our aim is to prove a multiplicity theorem for problem $(P)_\lambda$ describing the number of solutions as the parameter $\lambda > 0$ varies. Moreover, we provide sign information for all the solutions produced.

Problem $(P)_\lambda$ was first studied by Ambrosetti-Lupo [2], Ambrosetti-Mancini [3], and Struwe [32], [33, p. 147]. In all these works the differential operator involved is the Laplacian (i.e. $a(\xi) = \xi$ for all $\xi \in \mathbb{R}^N$) and the perturbation f is x-independent satisfying either $f \in C^1(\mathbb{R})$ (see [2], [3], [32]) or f is Lipschitz continuous (see [33]). Therein, it is proved that if $\lambda > \tilde{\lambda}_2$, where $\tilde{\lambda}_2$ denotes the second eigenvalue of $(-\Delta, H_0^1(\Omega))$, then the problem has at least three nontrivial solutions. However, they do not provide sign information for these solutions. An extension to the study of p-Laplacian equations ($1 < p < \infty$) was done by Averna-Motreanu-Marano [5], respectively, by Papageorgiou-Papageorgiou [29]. In the present paper we prove a similar three-solutions-theorem for problem $(P)_\lambda$ providing sign information for all solutions obtained. Moreover, in the particular case of $(p,2)$-equations (i.e., $a(\xi) = \|\xi\|^{p-2}\xi + \xi$ for all $\xi \in \mathbb{R}^N$ with $2 < p < \infty$ is the sum of a p-Laplacian and a Laplacian), we obtain the existence of four nontrivial solutions with complete sign information. In fact, as the results of Dancer [16] suggest (see also Struwe [33, p. 147]), one cannot expect more than four nontrivial solutions without any symmetry condition on the perturbation, even for $\lambda > 0$ large.

Our approach is variational based on the critical point theory coupled with the usage of suitable perturbation and truncation techniques and with comparison principles. In the last section we also employ Morse theory (critical groups) dealing with the special case of the $(p,2)$-equation. The main mathematical tools which will be used in this paper are recalled in the next section for the convenience of the reader.

2 Mathematical background and hypotheses

We start with some basic definitions and facts about critical point theory which we will need in the sequel. Let X be a Banach space and X^* its topological dual while $\langle \cdot, \cdot \rangle$ is taken for the duality brackets to the pair (X^*, X). We have the following definition.

Definition 2.1 *The functional $\varphi \in C^1(X)$ fulfills the Palais-Smale condition at the level $c \in \mathbb{R}$ (the PS_c-condition for short) if every sequence $(u_n)_{n\geq 1} \subseteq X$ satisfying $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ in X^*, admits a strongly convergent subsequence. We say that φ satisfies the Palais-Smale condition (the PS-condition for short) if it satisfies the PS_c-condition for every $c \in \mathbb{R}$.*

Using this compactness-type condition, we can prove the following minimax theorem known in the literature as the Mountain Pass Theorem.

Theorem 2.2 *If $\varphi \in C^1(X), u_1, u_2 \in X, \|u_2 - u_1\| > \rho > 0$,*

$$\max\{\varphi(u_1), \varphi(u_2)\} < \inf\{\varphi(u) : \|u - u_1\| = \rho\} =: \eta_\rho$$

and φ satisfies the PS_c-condition where

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} \varphi(\gamma(t))$$

with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_1, \gamma(1) = u_2\}$, then $c \ge \eta_\rho$ and c is a critical value of φ.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets

$$\begin{aligned} \varphi^c &= \{u \in X : \varphi(u) \le c\} && \text{(the sublevel set of } \varphi \text{ at } c), \\ K_\varphi &= \{u \in X : \varphi'(u) = 0\} && \text{(the critical set of } \varphi), \\ K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\} && \text{(the critical set of } \varphi \text{ at the level } c). \end{aligned}$$

Another result from critical point theory, which will be needed, is the so-called Second Deformation Lemma (see, for example, Gasiński-Papageorgiou [23, p. 628]).

Lemma 2.3 *If $\varphi \in C^1(X), a \in \mathbb{R}, a < b \le +\infty, \varphi$ satisfies the PS_c-condition for every $c \in [a, b)$, φ has no critical values in (a, b) and $\varphi^{-1}(a)$ contains at most a finite number of critical points of φ, then there exists a continuous map $h : [0,1] \times \left(\varphi^b \setminus K_\varphi^b\right) \to \varphi^b$ such that*

(a) $h(0, u) = u$ for all $u \in \varphi^b \setminus K_\varphi^b$;

(b) $h\left(1, \varphi^b \setminus K_\varphi^b\right) \subseteq \varphi^a$;

(c) $h(t, u) = u$ for all $(t, u) \in [0,1] \times \varphi^a$;

(d) $\varphi(h(t, u)) \le \varphi(h(s, u))$ for all $t, s \in [0,1], s \le t$, and all $u \in \varphi^b \setminus K_\varphi^b$.

Remark 2.4 In particular, Lemma 2.3 implies that φ^a is a strong deformation retract at $\varphi^b \setminus K_\varphi^b$.

As usual, we denote by $C_0^1(\overline{\Omega})$ the ordered Banach space

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\right\}$$

with positive cone

$$C_0^1(\overline{\Omega})_+ = \left\{u \in C_0^1(\overline{\Omega}) : u(x) \ge 0 \ \forall x \in \overline{\Omega}\right\},$$

which has a nonempty interior given by

$$\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) = \left\{u \in C_0^1(\overline{\Omega}) : u(x) > 0 \ \forall x \in \Omega \text{ and } \frac{\partial u}{\partial n}(x) < 0 \ \forall x \in \partial\Omega\right\},$$

where $n = n(x)$ is the outer unit normal at $x \in \partial\Omega$.

Let $g, h \in L^\infty(\Omega)$. We write $g \prec h$ if for every compact set $K \subseteq \Omega$ there exists $\varepsilon > 0$ such that $g(x) + \varepsilon \le h(x)$ for a.a. $x \in K$. Clearly, if $g, h \in C(\Omega)$ and $g(x) < h(x)$ for all $x \in \Omega$, then $g \prec h$.

In order to accommodate the extra linear term $-\Delta u$, an easy modification of the proof of Lemma 3.7 in Filippakis-O'Regan-Papageorgiou [20] (see also Arcoya-Ruiz [4, Proposition 2.6]) leads to the following strong comparison principle.

Proposition 2.5 *Let $\varepsilon \geq 0$, $p \leq r < \infty$, and $g, h \in L^\infty(\Omega)$. If $u, v \in C_0^1(\overline{\Omega})$ are solutions of*

$$\begin{aligned} -\Delta_p u - \Delta u + \varepsilon |u|^{r-2}u &= g \quad \text{in } \Omega, \\ -\Delta_p v - \Delta v + \varepsilon |v|^{r-2}v &= h \quad \text{in } \Omega, \end{aligned}$$

and $v \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, then $v - u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Now, we are ready to introduce the hypotheses on the map $a(\cdot)$. Let $\vartheta \in C^1(0, +\infty)$ be a function satisfying

$$0 < \hat{c} \leq \frac{t\vartheta'(t)}{\vartheta(t)} \leq c_0 \quad \text{and} \quad c_1 t^{p-1} \leq \vartheta(t) \leq c_2\left(1 + t^{p-1}\right) \tag{2.1}$$

for all $t > 0$ and with some constants $\hat{c}, c_0, c_1, c_2 > 0$.

Then the hypotheses on $a(\cdot)$ are the following.

H(a): $a(\xi) = a_0(\|\xi\|)\xi$ for all $\xi \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0, \infty)$, $t \mapsto ta_0(t)$ is strictly increasing, $\lim_{t \to 0^+} ta_0(t) = 0$, and $\lim\limits_{t \to 0^+} \dfrac{ta_0'(t)}{a_0(t)} > -1$;

(ii) $\|\nabla a(\xi)\| \leq c_3 \dfrac{\vartheta(\|\xi\|)}{\|\xi\|}$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and some $c_3 > 0$;

(iii) $(\nabla a(\xi)y, y)_{\mathbb{R}^N} \geq \dfrac{\vartheta(\|\xi\|)}{\|\xi\|}\|y\|^2$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and all $y \in \mathbb{R}^N$.

Remark 2.6 From hypothesis H(a)(i) we see at once that $a \in C^1\left(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^N\right) \cap C\left(\mathbb{R}^N, \mathbb{R}^N\right)$ implying that hypotheses H(a)(ii), (iii) make sense. Let $G_0(t) = \int_0^t sa_0(s)ds$ and consider the function $G : \mathbb{R}^N \to \mathbb{R}$ defined by $G(\xi) = G_0(\|\xi\|)$ for all $\xi \in \mathbb{R}^N$. Then

$$\nabla G(\xi) = G_0'(\|\xi\|)\frac{\xi}{\|\xi\|} = a_0(\|\xi\|)\xi = a(\xi) \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}.$$

Note that by virtue of hypothesis H(a)(i), we have $\nabla G(0) = 0$ as well. Evidently $G(\cdot)$ is convex and $G(0) = 0, \nabla G(\xi) = a(\xi)$ for all $\xi \in \mathbb{R}^N$. Hence we have the estimate

$$G(\xi) \leq (a(\xi), \xi)_{\mathbb{R}^N} \quad \text{for all } \xi \in \mathbb{R}^N. \tag{2.2}$$

The hypotheses H(a), (2.1) and the integral form of the mean value theorem lead to the following lemma, which summarizes the main properties of the map $a(\cdot)$.

Lemma 2.7 *Let the hypotheses H(a) be satisfied. Then, there hold*

(a) $\xi \to a(\xi)$ is maximal monotone and strictly monotone;

(b) $\|a(\xi)\| \leq c_4\left(1 + \|\xi\|^{p-1}\right)$ for all $\xi \in \mathbb{R}^N$ and some $c_4 > 0$;

(c) $(a(\xi), \xi)_{\mathbb{R}^N} \geq \frac{c_1}{p-1}\|\xi\|^p$ for all $\xi \in \mathbb{R}^N$.

As a direct consequence of Lemma 2.7 we obtain the following growth estimates for the potential $G(\cdot)$.

Corollary 2.8 *If hypotheses H(a) hold, then*

$$\frac{c_1}{p(p-1)}\|\xi\|^p \leq G(\xi) \leq c_5\,(1+\|\xi\|^p) \quad \textit{for all } \xi \in \mathbb{R}^N \textit{ and some } c_5 > 0.$$

Example 2.9 The following maps satisfy hypotheses H(a):

(a) Let $a(\xi) = \|\xi\|^{p-2}\xi$ with $1 < p < \infty$. This map corresponds to the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The potential is $G(\xi) = \frac{1}{p}\|\xi\|^p$ for all $\xi \in \mathbb{R}^N$.

(b) The function $a(\xi) = \|\xi\|^{p-2}\xi + \mu\|\xi\|^{q-2}\xi$ with $1 < q < p$ and $\mu > 0$ compares with the (p,q)-differential operator defined by $\Delta_p u + \mu\Delta_q u$ for all $u \in W_0^{1,p}(\Omega)$. The potential is $G(\xi) = \frac{1}{p}\|\xi\|^p + \frac{\mu}{q}\|\xi\|^q$ for all $\xi \in \mathbb{R}^N$. Equations driven by a (p,q)-differential operator arise in mathematical physics such as quantum physics (for existence of soliton solutions, see Benci-D'Avenia-Fortunato-Pisani [6]) and in plasma physics and biophysics (see Cherfils-Il′yasov [11]). Recently, there have been some existence and multiplicity results for such operators. We refer to the works of Cingolani-Degiovanni [12], Medeiros-Perera [27], Papageorgiou-Smyrlis [30], and Sun [34].

(c) If $a(\xi) = \left(1+\|\xi\|^2\right)^{\frac{p-2}{2}}\xi$ with $1 < p < \infty$, then this map represents the generalized p-mean curvature differential operator defined by

$$\operatorname{div}\left[(1+\|\nabla u\|^2)^{\frac{p-2}{2}}\nabla u\right] \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The potential is $G(\xi) = \frac{1}{p}\left[(1+\|\xi\|^2)^{\frac{p}{2}} - 1\right]$ for all $\xi \in \mathbb{R}^N$.

(d) Let $a(\xi) = \|\xi\|^{p-2}\xi \pm \frac{\|\xi\|^{p-2}\xi}{1+\|\xi\|^p}$ with $2 \leq p < \infty$. Then, the potential is given by $G(\xi) = \frac{1}{p}\|\xi\|^p \pm \frac{1}{p}\ln(1+\|\xi\|^p)$ for all $\xi \in \mathbb{R}^N$. It should be mentioned the work of Clément-García-Huidobro-Manásevich-Schmitt [14] where the authors also consider nonlinear nonhomogeneous equations using tools from the theory of Orlicz spaces producing solutions of constant sign for a different nonlinearity than ours.

Now, let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with subcritical growth in the second argument, that is

$$|f_0(x,s)| \leq a(x) + c|s|^{r-1} \quad \text{for a.a. } x \in \Omega, \text{ for all } s \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+$, $c > 0$, and $1 < r < p^*$, where p^* is the critical exponent of p given by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

Let $F_0(x,s) = \int_0^s f_0(x,t)dt$ and let $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\varphi_0(u) = \int_\Omega G(\nabla u)dx - \int_\Omega F_0(x,u)dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The following result is essentially due to Motreanu-Papageorgiou [28] (see also Brezis-Nirenberg [7], García Azorero-Peral Alonso-Manfredi [21] and Winkert [35] for earlier results in this direction). The result is a consequence of the nonlinear regularity theory (see [25], [26]).

Proposition 2.10 *Let the assumptions in H(a) be satisfied. If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, i.e., there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \textit{for all } h \in C_0^1(\overline{\Omega}) \textit{ with } \|h\|_{C_0^1(\overline{\Omega})} \leq \rho_0,$$

then $u_0 \in C_0^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ and u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, i.e., there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \textit{for all } h \in W_0^{1,p}(\Omega) \textit{ with } \|h\|_{W_0^{1,p}(\Omega)} \leq \rho_1.$$

Let $\frac{1}{p} + \frac{1}{p'} = 1$ and let $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be the nonlinear map defined by

$$\langle A(u), v\rangle = \int_\Omega (a(\nabla u), \nabla v)_{\mathbb{R}^N} dx \quad \text{for all } u, v \in W_0^{1,p}(\Omega). \tag{2.3}$$

The following result summarizes the basic properties of A (see, for example, Gasiński-Papageorgiou [22, p. 562]).

Proposition 2.11 *If hypotheses H(a) hold, then $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by (2.3) is continuous, monotone (hence maximal monotone), and of type $(S)_+$, i.e., if $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$.*

By Δ_r, $1 < r < \infty$, we denote the special case of the map above corresponding to the r-Laplacian, i.e.,

$$\langle \Delta_r(u), v\rangle = \int_\Omega \|\nabla u\|^{r-2}(\nabla u, \nabla v)_{\mathbb{R}^N} dx \quad \text{for all } u, v \in W_0^{1,r}(\Omega).$$

If $r = 2$, it reduces to the Laplacian and we write $\Delta_2 = \Delta \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$.

Next let us recall some basic facts about the spectrum of the negative Dirichlet r-Laplacian, i.e., of $\left(-\Delta_r, W_0^{1,r}(\Omega)\right)$ with $1 < r < \infty$. We consider the following nonlinear eigenvalue problem

$$\begin{aligned} -\Delta_r u &= \hat{\lambda}|u|^{r-2}u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.4}$$

A number $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of $\left(-\Delta_r, W_0^{1,r}(\Omega)\right)$ if problem (2.4) admits a nontrivial solution $\hat{u} \in W_0^{1,p}(\Omega)$ which is called an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. By $\hat{\sigma}(r)$ we denote the set of eigenvalues of $\left(-\Delta_r, W_0^{1,r}(\Omega)\right)$. It is known that the set $\hat{\sigma}(r)$ has a smallest element $\hat{\lambda}_1(r)$, which has the following properties:

- $\hat{\lambda}_1(r)$ is positive;

- $\hat{\lambda}_1(r)$ is isolated, that is, there exists $\varepsilon > 0$ such that $\left[\hat{\lambda}_1(r), \hat{\lambda}_1(r) + \varepsilon\right) \cap \hat{\sigma}(r) = \hat{\lambda}_1(r)$;
- $\hat{\lambda}_1(r)$ is simple, that is, if u, v are two eigenfunctions corresponding to $\hat{\lambda}_1(r)$, then $u = kv$ for some $k \in \mathbb{R}$;
-

$$\hat{\lambda}_1(r) = \inf\left[\frac{\|\nabla u\|^r_{L^r(\Omega)}}{\|u\|^r_{L^r(\Omega)}} : u \in W^{1,r}_0(\Omega), u \neq 0\right]. \tag{2.5}$$

In (2.5) the infimum is realized on the corresponding one dimensional eigenspace. Regarding (2.5) it is also clear that the eigenfunctions corresponding to $\hat{\lambda}_1(r)$ do not change sign. In fact it turns out that $\hat{\lambda}(r)$ is the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have eigenfunctions which are nodal (i.e., sign changing). In what follows by $\hat{u}_1(r)$ we denote the L^r-normalized (i.e., $\|\hat{u}_1(r)\|_{L^r(\Omega)} = 1$) positive eigenfunction corresponding to $\hat{\lambda}_1(r)$. The nonlinear regularity theory and the nonlinear maximum principle imply that $\hat{u}_1(r) \in \operatorname{int}\left(C^1_0(\overline{\Omega})_+\right)$ (see [25], [26] and [23, pp. 737–738]). Furthermore, the set $\hat{\sigma}(r)$ is closed and since $\hat{\lambda}_1(r)$ is isolated the second eigenvalue $\hat{\lambda}_2(r)$ is well-defined by

$$\hat{\lambda}_2(r) = \inf\left[\hat{\lambda} \in \hat{\sigma}(r) : \hat{\lambda} > \hat{\lambda}_1(r)\right].$$

Now, let $\partial B^{L^r}_1 = \{u \in L^r(\Omega) : \|u\|_{L^r(\Omega)} = 1\}$ and $M_r = W^{1,r}_0(\Omega) \cap \partial B^{L^r}_1$. Then, $\hat{\lambda}_2(r)$ admits the following variational characterization (see Cuesta-de Figueiredo-Gossez [15]).

Proposition 2.12 *There holds*

$$\hat{\lambda}_2(r) = \inf_{\hat{\gamma} \in \hat{\Gamma}(r)} \max_{-1 \leq t \leq 1} \|\nabla \hat{\gamma}(t)\|^r_{L^r(\Omega)},$$

where $\hat{\Gamma}(r) = \{\hat{\gamma} \in C([-1, 1], M_r) : \hat{\gamma}(-1) = -\hat{u}_1(r), \hat{\gamma}(1) = \hat{u}_1(r)\}$.

The Lusternik-Schnirelmann minimax scheme gives a whole strictly increasing sequence $\left(\hat{\lambda}_k(r)\right)_{k \geq 1}$ of eigenvalues, but it is not known if this sequence exhausts the whole set $\hat{\sigma}(r)$. Indeed, this is true if $N = 1$ (ordinary differential equations) or if $r = 2$ (linear eigenvalue problem). In the latter case by $E\left(\hat{\lambda}_k(2)\right), k \geq 1$, we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k(2), k \geq 1$.

Next, let us recall some basic facts about Morse theory. Let X be a Banach space and let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, the term $H_k(Y_1, Y_2)$ stands for the $k \overset{th}{=}$-relative singular homology group with integer coefficients. For $k < 0$, we have $H_k(Y_1, Y_2) = 0$. Let $\varphi \in C^1(X)$ and let $u \in X$ be an isolated critical point of φ with $\varphi(u) = c$ (i.e., $u \in K^c_\varphi$). The critical groups of φ at $u \in K^c_\varphi$ are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \geq 0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that this definition of critical groups is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the PS-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \geq 0.$$

This definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$ which is a consequence of the Second Deformation Lemma stated in Lemma 2.3.

We assume that K_φ is finite and introduce the following series in $t \in \mathbb{R}$:

$$M(t,u) = \sum_{k\geq 0} \operatorname{rank} C_k(\varphi,u)t^k \quad \text{for all } u \in K_\varphi,$$
$$P(t,\infty) = \sum_{k\geq 0} \operatorname{rank} C_k(\varphi,\infty)t^k.$$

Then, the Morse relation reads as follows:

$$\sum_{u\in K_\varphi} M(t,u) = P(t,\infty) + (1+t)Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{2.6}$$

where $Q(t)$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

It is well-known that if a functional satisfies the PS-condition or the C-condition and it is bounded below, then it is coercive (see Čaklović-Li-Willem [9] and Gasiński-Papageorgiou [23, p. 614]). The converse is in general not true. However, in the setting of this work the converse is true.

More precisely, let $\hat{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$\left|\hat{f}(x,s)\right| \leq a(x)\left(1 + |s|^{r-1}\right) \quad \text{for a.a. } x \in \Omega, \text{ for all } s \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+$, and $p \leq r < p^*$. We set $\hat{F}(x,s) = \int_0^s \hat{f}(x,t)dt$ and consider the C^1-functional $\hat{\varphi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}(u) = \int_\Omega G(\nabla u)dx - \int_\Omega \hat{F}(x,u)dx.$$

Proposition 2.13 *If $\hat{\varphi}$ is coercive, then it satisfies the PS-condition.*

Proof. Suppose $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is a PS-sequence, that is

$$|\hat{\varphi}(u_n)| \leq \tilde{M} \quad \text{for some } \tilde{M} > 0, \text{ for all } n \geq 1, \tag{2.7}$$
$$\hat{\varphi}'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega). \tag{2.8}$$

The statement in (2.7) along with the coercivity of $\hat{\varphi}$ implies that $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. Therefore, we may assume that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\Omega). \tag{2.9}$$

From (2.8) it follows

$$\left|\langle A(u_n), v\rangle - \int_\Omega \hat{f}(x,u_n)vdx\right| \leq \frac{\varepsilon_n \|v\|_{W_0^{1,p}(\Omega)}}{1 + \|u_n\|_{W_0^{1,p}(\Omega)}},$$

for all $v \in W_0^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$. Now, choosing $v = u_n - u$, passing to the limit as $n \to \infty$, and using the convergence properties in (2.9) we obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,$$

which by the $(S)_+$-property of A (see Proposition 2.11) gives $u_n \to u$ in $W_0^{1,p}(\Omega)$. Hence, $\hat{\varphi}$ fulfills the PS-condition. □

Finally we conclude this section by fixing our notation. Throughout this paper we denote the norm of $W_0^{1,p}(\Omega)$ through $\|\cdot\|_{W_0^{1,p}(\Omega)}$ and thanks to Poincare's inequality we have $\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}$ for all $u \in W_0^{1,p}(\Omega)$. The norm of $\mathbb{R}^N$ is denoted by $\|\cdot\|$ and $(\cdot,\cdot)_{\mathbb{R}^N}$ stands for the inner product in $\mathbb{R}^N$. For $s \in \mathbb{R}$, we set $s^{\pm} = \max\{\pm s, 0\}$ and for $u \in W_0^{1,p}(\Omega)$ we define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. It is well known that

$$u^{\pm} \in W_0^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

The Lebesgue measure on $\mathbb{R}^N$ is given by $|\cdot|_N$. Finally, for any Carathéodory function $h : \Omega\times\mathbb{R} \to \mathbb{R}$, we define the Nemytskij operator corresponding to the function h by

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

3 Three solutions

This section is devoted to the study of problem $(P)_\lambda$ with $\lambda > 0$ appropriately large. We prove the existence of at least three nontrivial smooth solutions including complete sign information for these solutions. Precisely, it will be shown that the first solution is positive, the second one is negative, and the last one has changing sign (nodal).

Before we start with our results we state some stronger hypotheses on the map $a(\cdot)$ as in H(a) which will be needed in our proofs.

H(a)': $a(\xi) = a_0(\|\xi\|)\xi$ for all $\xi \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$, hypotheses H(a)'(i)–(iii) are the same as the corresponding hypotheses H(a)(i)–(iii) and

(iv) if $G_0(t) = \int_0^t s a_0(s)ds$ for all $t > 0$, then $t \mapsto G_0\left(t^{\frac{1}{q}}\right)$ is convex in $(0, +\infty)$ and

$$\limsup_{t\to 0^+} \frac{G_0(t)}{t^q} < +\infty.$$

Remark 3.1 The examples presented in Section 2 still satisfy hypotheses H(a)'. Note that hypothesis H(a)'(iv) implies

$$G(\xi) \le c_6 \left(\|\xi\|^q + \|\xi\|^p\right) \quad \text{for all } \xi \in \mathbb{R}^N, \tag{3.1}$$

with some $c_6 > 0$.

The hypotheses on the perturbation f are the following:

H(f)$_1$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(x, s)| \le a_\rho(x) \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \le \rho;$$

(ii) $\lim_{s\to\pm\infty} \frac{f(x, s)}{|s|^{p-2}s} = +\infty$ uniformly for a.a. $x \in \Omega$;

(iii) $\lim_{s\to 0} \frac{f(x, s)}{|s|^{q-2}s} = 0$ uniformly for a.a. $x \in \Omega$.

Remark 3.2 We point out that no growth restriction is imposed on $f(x,\cdot)$. This is in contrast to the considerations in [5] and [29], where it is required that $f(x,\cdot)$ has subcritical growth for a.a. $x \in \Omega$.

Example 3.3 The following functions satisfy hypotheses $H(f)_1$:

$$f_1(x,s) = a(x)|s|^{r-2}s \qquad \text{and} \qquad f_2(x,s) = a(x)|s|^{p-2}s\ln(1+|s|)$$

with $a \in L^\infty(\Omega)$ and $p < r < \infty$.

Given any $\eta > 0$, by virtue of hypothesis H(f)$_1$(ii), there exists $M_1 = M_1(\eta) > 1$ such that

$$f(x,s)s \geq \eta|s|^p \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \geq M_1. \tag{3.2}$$

Let $\overline{u} \equiv \varsigma \in [M_1, +\infty)$. Taking into account (3.2), $q \leq p$, and $M_1 > 1$, we conclude for $\eta = \lambda$

$$0 \geq \lambda\overline{u}^{q-1} - f(x,\overline{u}) \quad \text{for a.a. } x \in \Omega. \tag{3.3}$$

Similarly, if $\underline{v} \equiv -\varsigma$, then

$$0 \leq \lambda|\underline{v}|^{q-2}\underline{v} - f(x,\underline{v}) \quad \text{for a.a. } x \in \Omega.$$

Applying $\overline{u}$ and $\underline{v}$, we introduce the following truncations of the reaction in problem $(P)_\lambda$:

$$h^+_\lambda(x,s) = \begin{cases} 0 & \text{if } s < 0, \\ \lambda s^{q-1} - f(x,s) & \text{if } 0 \leq s \leq \overline{u}, \\ \lambda\overline{u}^{q-1} - f(x,\overline{u}) & \text{if } \overline{u} < s, \end{cases} \tag{3.4}$$

and

$$h^-_\lambda(x,s) = \begin{cases} \lambda|\underline{v}|^{q-2}\underline{v} - f(x,\underline{v}) & \text{if } s < \underline{v}, \\ \lambda|s|^{q-2}s - f(x,s) & \text{if } \underline{v} \leq s \leq 0, \\ 0 & \text{if } 0 < s. \end{cases}$$

It is easy to verify that both mappings are Carathéodory functions. We set $H^\pm_\lambda(x,s) = \int_0^s h^\pm_\lambda(x,t)dt$ and consider the C^1-functionals $\hat{\varphi}^\pm_\lambda : W^{1,p}_0(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}^\pm_\lambda(u) = \int_\Omega G(\nabla u)dx - \int_\Omega H^\pm_\lambda(x,u)dx.$$

Now, we are ready to produce two nontrivial constant sign solutions of $(P)_\lambda$ for suitable $\lambda > 0$ large enough. In what follows, we use the notation

$$[0,\overline{u}] = \left\{u \in W^{1,p}_0(\Omega) : 0 \leq u(x) \leq \overline{u} \text{ a.e. in } \Omega\right\},$$
$$[\underline{v},0] = \left\{u \in W^{1,p}_0(\Omega) : \underline{v} \leq u(x) \leq 0 \text{ a.e. in } \Omega\right\}.$$

Proposition 3.4 *Suppose that hypotheses H(a)' and H(f)$_1$ are fulfilled and assume that*

$$\lambda > \begin{cases} qc_6\hat{\lambda}_1(q) & \text{if } q < p, \\ 2qc_6\hat{\lambda}_1(q) & \text{if } q = p \end{cases}$$

with the positive constant c_6 of Remark 3.1. Then problem $(P)_\lambda$ has at least two nontrivial constant sign solutions

$$u_0 \in [0,\overline{u}] \cap \operatorname{int}\left(C^1_0(\overline{\Omega})_+\right) \quad \text{and} \quad v_0 \in [\underline{v},0] \cap \left(-\operatorname{int}\left(C^1_0(\overline{\Omega})_+\right)\right).$$

Proof. We start with the existence of the positive solution. Thanks to the truncation in (3.4) and Corollary 2.8, it is obvious that $\hat{\varphi}_\lambda^+$ is coercive and taking into account the Sobolev embedding theorem we verify that $\hat{\varphi}_\lambda^+$ is sequentially weakly lower semicontinuous as well. Hence, by virtue of the Weierstrass theorem, we find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda^+(u_0) = \inf\left[\hat{\varphi}_\lambda^+(u) : u \in W_0^{1,p}(\Omega)\right] = \hat{m}_\lambda^+. \tag{3.5}$$

Given $\varepsilon > 0$, from hypothesis H(f)(iii), we can find $\delta = \delta(\varepsilon) \in (0, \overline{u})$ such that

$$F(x,s) \le \frac{\varepsilon}{q}|s|^q \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \le \delta. \tag{3.6}$$

Recalling $\hat{u}_1(q) \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, we choose $t \in (0,1)$ such that $t\hat{u}_1(q)(x) \in [0,\delta]$ for all $x \in \overline{\Omega}$. Then, due to (3.1), (3.4), (3.6) combined with $\|\hat{u}_1(q)\|_{L^q(\Omega)} = 1$ and the fact that $\delta < \overline{u}$, we obtain

$$\begin{aligned}
\hat{\varphi}_\lambda^+(t\hat{u}_1(q)) &= \int_\Omega G(\nabla(t\hat{u}_1(q)))dx - \int_\Omega H_\lambda^+(x, t\hat{u}_1(q))dx \\
&= \int_\Omega G(\nabla(t\hat{u}_1(q)))dx - \frac{\lambda t^q}{q}\|\hat{u}_1(q)\|_{L^q(\Omega)}^q + \int_\Omega F(x, t\hat{u}_1(q))dx \\
&\le c_6\left(t^q\|\nabla(\hat{u}_1(q))\|_{L^q(\Omega)}^q + t^p\|\nabla(\hat{u}_1(q))\|_{L^p(\Omega)}^p\right) - \frac{\lambda t^q}{q} + \frac{\varepsilon t^q}{q} \\
&= c_6\left(t^q\hat{\lambda}_1(q) + t^p\|\nabla(\hat{u}_1(q))\|_{L^p(\Omega)}^p\right) - \frac{\lambda-\varepsilon}{q}t^q \\
&= t^q\left[\frac{qc_6\hat{\lambda}_1(q) + \varepsilon - \lambda}{q}\right] + c_6t^p\|\nabla(\hat{u}_1(q))\|_{L^p(\Omega)}^p.
\end{aligned} \tag{3.7}$$

Now, we choose ε such that

$$0 < \varepsilon < \begin{cases} \lambda - qc_6\hat{\lambda}_1(q) & \text{if } q < p, \\ \lambda - 2qc_6\hat{\lambda}_1(q) & \text{if } q = p. \end{cases}$$

In both cases the right-hand side of (3.7) can be estimated above by zero. This gives (see also (3.5))

$$\hat{\varphi}_\lambda^+(u_0) = \hat{m}_\lambda^+ < 0 = \hat{\varphi}_\lambda^+(0),$$

which means that $u_0 \neq 0$ is nontrivial. Since u_0 is the global minimum of $\hat{\varphi}_\lambda^+$, we have $\left(\hat{\varphi}_\lambda^+\right)'(u_0) = 0$ which results in

$$A(u_0) = N_{h_\lambda^+}(u_0). \tag{3.8}$$

Taking $-u_0^- \in W_0^{1,p}(\Omega)$ as test function in (3.8) and applying Lemma 2.7(c) and (3.4), it follows

$$\frac{c_1}{p-1}\|\nabla u_0^-\|_{L^p(\Omega)}^p \le 0$$

ensuring that $u_0 \ge 0$. Next, we choose $(u_0 - \overline{u})^+ \in W_0^{1,p}(\Omega)$ as test function in (3.8). With the aid of (3.3) and (3.4), we get

$$\begin{aligned}
\langle A(u_0), (u_0-\overline{u})^+\rangle &= \int_\Omega h_\lambda^+(x,u_0)(u_0-\overline{u})^+dx \\
&= \int_\Omega \lambda\overline{u}^{q-1}(u_0-\overline{u})^+dx - \int_\Omega f(x,\overline{u})(u_0-\overline{u})^+dx \\
&\le 0.
\end{aligned}$$

From this we see that

$$\int_{\{u_0>\overline{u}\}} (a(\nabla u_0), \nabla u_0)_{\mathbb{R}^N} dx \leq 0$$

meaning that, in combination with Lemma 2.7(c),

$$\frac{c_1}{p-1} \int_{\{u_0>\overline{u}\}} \|\nabla u_0\|^p dx \leq 0.$$

We conclude that $|\{u_0 > \overline{u}\}|_N = 0$, thus $u_0 \leq \overline{u}$. Recalling that $u_0 \geq 0$ we get $u_0 \in [0, \overline{u}]$ with $u_0 \not\equiv 0$. Then, thanks to the definition of the truncation function in (3.4), relation (3.8) becomes

$$A(u_0) = \lambda u_0^{q-1} - N_f(u_0)$$

which can equivalently be written as

$$\begin{aligned} -\operatorname{div} a(\nabla u_0) &= \lambda u_0^{q-1} - f(x, u_0) && \text{in } \Omega, \\ u_0 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Since $u_0 \in L^\infty(\Omega)$ (see Ladyzhenskaya-Ural′tseva [25, p. 286]) we can apply the regularity results of Lieberman [26, p. 320] to obtain that $u_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$. Given $\varepsilon \in (0, \lambda)$, owing to hypotheses $\mathrm{H(f)}_1$(i),(iii), we find a constant $c_\varepsilon > 0$ such that

$$f(x, s) \leq \varepsilon s^{q-1} + c_\varepsilon s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \leq s \leq \overline{u}.$$

Hence, one has

$$\operatorname{div} a(\nabla u_0(x)) \leq c_\varepsilon u_0(x)^{p-1} \text{ a.e. in } \Omega. \tag{3.9}$$

Then, the strong maximum principle of Pucci-Serrin [31, Theorem 2.5.1] yields $u_0(x) > 0$ for all $x \in \Omega$. Applying the Boundary Point Lemma (see again Pucci-Serrin [31, Theorem 5.5.1]) gives $u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Similarly, working with $\hat{\varphi}_\lambda^-$ instead of $\hat{\varphi}_\lambda^+$, we establish the existence of a nontrivial negative solution $v_0 \in [\underline{v}, 0] \cap \left(-\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)\right)$. □

Our next proceeding is the proof of the existence of extremal solutions of $(P)_\lambda$ with $\lambda > 0$ large enough as before. That means, we prove the existence of a smallest positive solution and a greatest negative solution, both of them are nontrivial and smooth.

To this end, let $S_+(\lambda)$ be the set containing all nontrivial positive solutions of problem $(P)_\lambda$. As shown in Filippakis-Kristály-Papageorgiou [19, p. 431], exploiting the monotonicity of $a(\cdot)$ (see Lemma 2.7(a)), we have that $S_+(\lambda)$ is downward directed, i.e., if $u_1, u_2 \in S_+(\lambda)$, then there is an element $\tilde{u} \in S_+(\lambda)$ such that $\tilde{u} \leq u_1, \tilde{u} \leq u_2$. Therefore, without loss of generality, we can restrict our treatment to the set

$$\hat{S}_+(\lambda) = S_+(\lambda) \cap [0, \overline{u}].$$

Likewise, if $S_-(\lambda)$ is the set of all nontrivial negative solutions of problem $(P)_\lambda$, we can focus on the set

$$\hat{S}_-(\lambda) = S_-(\lambda) \cap [\underline{v}, 0].$$

By means of Proposition 3.4, we know that both sets are nonempty, i.e., $\hat{S}_+(\lambda) \neq \emptyset$ and $\hat{S}_-(\lambda) \neq \emptyset$. Additionally, by truncating $f(x,\cdot)$ at $\underline{v}$ (from below) and $\overline{u}$ (from above), we may assume without loss of generality that

$$|f(x,s)| \leq \hat{a}(x) \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R} \text{ with } \hat{a} \in L^\infty(\Omega)_+$$

(see hypothesis H(f)$_1$(i)). Using this observation and hypothesis H(f)$_1$(iii), we see that for a given $\varepsilon > 0$ and $r \in (p, p^*)$, we can find $c_7 = c_7(\varepsilon, r) > 0$ such that

$$f(x,s)s \leq \varepsilon|s|^q + c_7|s|^r \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R}. \tag{3.10}$$

Let $\lambda > 0$ and $\varepsilon \in (0, \lambda)$. We consider the following auxiliary Dirichlet problem

$$\begin{aligned} -\operatorname{div} a(\nabla u) &= (\lambda - \varepsilon)|u|^{q-2}u - c_7|u|^{r-2}u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.11}$$

Our next result is about the uniqueness of constant sign solutions of problem (3.11).

Proposition 3.5 *If hypotheses H(a)' hold and if*

$$\lambda > \begin{cases} qc_6\hat{\lambda}_1(q) & \text{if } q < p, \\ 2qc_6\hat{\lambda}_1(q) & \text{if } q = p \end{cases}$$

is satisfied, then problem (3.11) has a unique nontrivial positive solution $u_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and by virtue of the oddness of (3.11), $v_* = -u_* \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is the unique nontrivial negative solution.

Proof. First, we establish the existence of a nontrivial positive solution for problem (3.11). To this end, we consider the C^1-functional $\psi_\lambda^+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda^+(u) = \int_\Omega G(\nabla u)dx - \frac{\lambda - \varepsilon}{q}\|u^+\|_{L^q(\Omega)}^q + \frac{c_7}{r}\|u^+\|_{L^r(\Omega)}^r.$$

Since $q \leq p < r$ and due to Corollary 2.8, we easily see that ψ_λ^+ is coercive. In addition, one verifies its property to be sequentially weakly lower semicontinuous. Hence, its global minimizer exists, namely $u_* \in W_0^{1,p}(\Omega)$ satisfies

$$\psi_\lambda^+(u_*) = \inf\left[\psi_\lambda^+(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{3.12}$$

As in the proof of Proposition 3.4, the choice of $\lambda > 0$ leads to

$$\psi_\lambda^+(u_*) < 0 = \psi_\lambda^+(0)$$

meaning that $u_* \neq 0$. From (3.12) we obtain $\left(\psi_\lambda^+\right)'(u_*) = 0$ which results in

$$A(u_*) = (\lambda - \varepsilon)\left(u_*^+\right)^{q-1} - c_7\left(u_*^+\right)^{r-1}. \tag{3.13}$$

Acting on (3.13) with $-u_*^- \in W_0^{1,p}(\Omega)$ and using Lemma 2.7, we check at once that $u_* \geq 0$ and $u_* \neq 0$. Then, (3.13) becomes

$$A(u_*) = (\lambda - \varepsilon)u_*^{q-1} - c_7u_*^{r-1}.$$

Hence, u_* is a nontrivial positive solution of (3.11). Moreover, as before (see the proof of Proposition 3.4) using the nonlinear regularity theory (see [25], [26]) and the nonlinear maximum principle (see [31]), it holds $u_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

It remains to show the uniqueness of this solution u_*. For this purpose, we consider the integral functional $\sigma_+ : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ defined by

$$\sigma_+(u) = \begin{cases} \displaystyle\int_\Omega G\left(\nabla u^{\frac{1}{q}}\right) dx & \text{if } u \geq 0, u^{\frac{1}{q}} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \operatorname{dom}\sigma_+$ and let $u = (tu_1 + (1-t)u_2)^{\frac{1}{q}}$ with $t \in [0,1]$. From Díaz-Saá [17, Lemma 1] we have

$$\|\nabla u(x)\| \leq \left(t\left\|\nabla u_1(x)^{\frac{1}{q}}\right\|^q + (1-t)\left\|\nabla u_2(x)^{\frac{1}{q}}\right\|^q\right)^{\frac{1}{q}} \quad \text{a.e. in } \Omega,$$

thus,

$$\begin{aligned} G_0\left(\|\nabla u(x)\|\right) &\leq G_0\left(\left(t\left\|\nabla u_1(x)^{\frac{1}{q}}\right\|^q + (1-t)\left\|\nabla u_2(x)^{\frac{1}{q}}\right\|^q\right)^{\frac{1}{q}}\right) \\ &\leq tG_0\left(\left\|\nabla u_1(x)^{\frac{1}{q}}\right\|\right) + (1-t)G_0\left(\left\|\nabla u_2(x)^{\frac{1}{q}}\right\|\right), \end{aligned}$$

thanks to hypothesis H(a)(iv) and the fact that G_0 is increasing. Since $G(\xi) = G_0(\|\xi\|)$ for all $\xi \in \mathbb{R}^N$, it follows that

$$G(\nabla u(x)) \leq tG\left(\nabla u_1(x)^{\frac{1}{q}}\right) + (1-t)G\left(\nabla u_2(x)^{\frac{1}{q}}\right) \quad \text{a.e. in } \Omega$$

proving that σ_+ is convex. Using Fatou's Lemma we infer that σ_+ is lower semicontinuous as well.

Let $u, v \in W_0^{1,p}(\Omega)$ be two nontrivial positive solutions of (3.11). As done in the first part of the proof, we know that $u, v \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, thus $u, v \in \operatorname{dom}\sigma_+$. Let $h \in C_0^1(\overline{\Omega})$. For $t \in (0,1)$ sufficiently small we have $u^q + th, v^q + th \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ which implies that σ_+ is Gateaux differentiable at u^q and v^q in the direction h. Moreover, via the chain rule, we obtain

$$\sigma'_+\left(u^q\right)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(\nabla u)}{u^{q-1}} h dx, \tag{3.14}$$

$$\sigma'_+\left(v^q\right)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(\nabla v)}{v^{q-1}} h dx. \tag{3.15}$$

Since σ_+ is convex, it is clear that σ'_+ is monotone. Therefore, using (3.14) and (3.15), it holds

$$\begin{aligned} 0 &\leq \left\langle \sigma'_+\left(u^q\right) - \sigma'_+\left(v^q\right), u^q - v^q \right\rangle_{L^1(\Omega)} \\ &= \frac{1}{q}\int_\Omega \left(\frac{-\operatorname{div} a(\nabla u)}{u^{q-1}} + \frac{\operatorname{div} a(\nabla v)}{v^{q-1}}\right)\left(u^q - v^q\right) dx \\ &= \frac{1}{q}\int_\Omega \left(\frac{(\lambda-\varepsilon)u^{q-1} - c_7 u^{r-1}}{u^{q-1}} - \frac{(\lambda-\varepsilon)v^{q-1} - c_7 v^{r-1}}{v^{q-1}}\right)\left(u^q - v^q\right) dx \\ &= \frac{c_7}{q}\int_\Omega \left(v^{r-q} - u^{r-q}\right)\left(u^q - v^q\right) dx. \end{aligned}$$

Since $s \mapsto s^{r-q}$ is strictly increasing in $(0,\infty)$, it follows that $u = v$, hence $u_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is the unique nontrivial positive solution of (3.11).

Evidently the oddness of (3.11) implies that $v_* = -u_* \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is the unique nontrivial negative solution of (3.11). □

Now, these two solutions $u_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $v_* = -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ of the auxiliary problem (3.11) lead to the existence of extremal nontrivial constant sign solutions for problem $(P)_\lambda$ when $\lambda > 0$ is sufficiently large. We obtain the following result.

Proposition 3.6 *Let hypotheses H(a)' and H(f)$_1$ hold and suppose*

$$\lambda > \begin{cases} qc_6\hat{\lambda}_1(q) & \text{if } q < p, \\ 2qc_6\hat{\lambda}_1(q) & \text{if } q = p. \end{cases}$$

Then problem $(P)_\lambda$ has a smallest nontrivial positive solution $u_+ \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and a greatest nontrivial negative solution $v_- \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Proof. As we already noted, it suffices to establish the existence of a smallest nontrivial element in $\hat{S}_+(\lambda) = S_+(\lambda) \cap [0,\overline{u}] \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and the existence of a greatest nontrivial negative solution in $\hat{S}_-(\lambda) = S_-(\lambda) \cap [\underline{v},0] \subseteq -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Claim: $u_* \le u$ for all $u \in \hat{S}_+(\lambda)$.

Let $\tilde{u} \in \hat{S}_*(\lambda)$ and consider the Carathéodory function

$$\gamma_\lambda^+(x,s) = \begin{cases} 0 & \text{if } s < 0, \\ (\lambda-\varepsilon)s^{q-1} - c_7 s^{r-1} & \text{if } 0 \le s \le \tilde{u}(x), \\ (\lambda-\varepsilon)\tilde{u}(x)^{q-1} - c_7\tilde{u}(x)^{r-1} & \text{if } \tilde{u}(x) < s. \end{cases} \tag{3.16}$$

We set $\Gamma_\lambda^+(x,s) = \int_0^s \gamma_\lambda^+(x,t)dt$ and consider the C^1-functional $\hat{\psi}_\lambda^+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\lambda^+(u) = \int_\Omega G(\nabla u)dx - \int_\Omega \Gamma_\lambda^+(x,u)dx.$$

Thanks to (3.16) it is obvious that $\hat{\psi}_\lambda^+$ is coercive and it is also sequentially weakly lower semicontinuous. Thus, we find $\hat{u}_* \in W_0^{1,p}(\Omega)$ such that

$$\hat{\psi}_\lambda^+(\hat{u}_*) = \inf\left[\hat{\psi}_\lambda^+(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{3.17}$$

As before (see the proof of Proposition 3.4 and recall the choice of $\lambda > 0$), we have

$$\hat{\psi}_\lambda^+(\hat{u}_*) < 0 = \hat{\psi}_\lambda^+(0),$$

hence $\hat{u}_* \neq 0$. From (3.17) we have

$$A(\hat{u}_*) = N_{\gamma_\lambda^+}(\hat{u}_*). \tag{3.18}$$

Applying again $-\hat{u}_*^- \in W_0^{1,p}(\Omega)$ as test function to (3.18), it follows from Lemma 2.7(c) that $\hat{u}_* \ge 0, \hat{u}_* \neq 0$. Furthermore, we act on (3.18) with $(\hat{u}_* - \tilde{u})^+ \in W_0^{1,p}(\Omega)$. Then, from (3.10), (3.16) and

the fact that $\tilde{u} \in \hat{S}_+(\lambda)$, it follows

$$\begin{aligned}\langle A(\hat{u}_*), (\hat{u}_* - \tilde{u})^+ \rangle &= \int_\Omega \gamma_\lambda^+(x, \hat{u}_*)(\hat{u}_* - \tilde{u})^+ \, dx \\ &= \int_\Omega \left((\lambda - \varepsilon)\tilde{u}^{q-1} - c_7\tilde{u}^{r-1}\right)(\hat{u}_* - \tilde{u})^+ \, dx \\ &\leq \int_\Omega \left(\lambda\tilde{u}^{q-1} - f(u, \tilde{u})\right)(\hat{u}_* - \tilde{u})^+ \, dx \\ &= \langle A(\tilde{u}), (\hat{u}_* - \tilde{u})^+ \rangle.\end{aligned}$$

We deduce

$$\int_{\{\hat{u}_* > \tilde{u}\}} (a(\nabla \hat{u}_*) - a(\nabla \tilde{u}), \nabla \hat{u}_* - \nabla \tilde{u})_{\mathbb{R}^N} \, dx \leq 0,$$

which implies that $|\{\hat{u}_* > \tilde{u}\}|_N = 0$ due to Lemma 2.7(a). Hence

$$0 \neq \hat{u}_* \in [0, \tilde{u}] = \left\{u \in W_0^{1,p}(\Omega) : 0 \leq u(x) \leq \tilde{u}(x) \text{ a.e. in } \Omega\right\}.$$

By virtue of (3.16) and (3.18), we see that $\hat{u}_*$ is a nontrivial positive solution of (3.11), hence $\hat{u}_* = u_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ (see Proposition 3.5). Therefore $u_* \leq \tilde{u}$ and since $\tilde{u} \in \hat{S}_+(\lambda)$ is arbitrary, we have proved the claim.

Let $C \subseteq \hat{S}_+(\lambda)$ be a chain, i.e., a totally ordered subset of $\hat{S}_+(\lambda)$. According to Dunford-Schwartz [18, p. 336], there exists a sequence $(u_n)_{n\geq 1} \subseteq \hat{S}_+(\lambda)$ such that

$$\inf C = \inf_{n\geq 1} u_n.$$

We have

$$A(u_n) = \lambda u_n^{q-1} - N_f(u_n), \qquad u_* \leq u_n \leq \overline{u} \quad \text{for all } n \geq 1. \tag{3.19}$$

Therefore, $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded and we may assume that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega). \tag{3.20}$$

Acting on (3.19) with the test function $u_n - u \in W_0^{1,p}(\Omega)$ and passing to the limit as $n \to \infty$ combined with (3.20), we obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u \rangle = 0. \tag{3.21}$$

By means of Proposition 2.11, we know that A fulfills the (S_+)-property which in view of (3.21) and the weak convergence in $W_0^{1,p}(\Omega)$ gives $u_n \to u$ in $W_0^{1,p}(\Omega)$. Then, we directly obtain from (3.19) that

$$A(u) = \lambda u^{q-1} - N_f(u), \qquad u_* \leq u \leq \overline{u}$$

meaning that $u \in \hat{S}_+(\lambda)$ and $u = \inf C$. Since C is an arbitrary chain in $\hat{S}_+(\lambda)$, the Kuratowski-Zorn Lemma implies that $\hat{S}_+(\lambda)$ has a minimal element $u_+ \in \hat{S}_+(\lambda)$. Since $\hat{S}_+(\lambda)$ is downward directed, we conclude that u_+ is the smallest nontrivial positive solution of $(P)_\lambda$.

Similarly, working with the set $\hat{S}_-(\lambda)$ and using again the Kuratowski-Zorn Lemma, we obtain that $v_- \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is the greatest nontrivial negative solution of $(P)_\lambda$. □

Finally, we will prove the existence of a sign-changing solution y_0 of $(P)_\lambda$ lying between these two extremal nontrivial constant sign solutions obtained in the last proposition.

Proposition 3.7 *If hypotheses H(a)' and H(f)$_1$ hold and if*

$$\lambda > \begin{cases} qc_6\hat{\lambda}_2(q) & \text{if } q < p, \\ 2qc_6\hat{\lambda}_2(q) & \text{if } q = p, \end{cases}$$

is satisfied, then problem $(P)_\lambda$ has a nodal solution $y_0 \in C_0^1\left(\overline{\Omega}\right)$.

Proof. Let $u_+ \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $v_- \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ be the two extremal nontrivial constant sign solutions of $(P)_\lambda$ produced in Proposition 3.6. We introduce the following truncation function of the reaction of problem $(P)_\lambda$

$$h_\lambda(x,s) = \begin{cases} \lambda\left|v_-(x)\right|^{q-2}v_-(x) - f\left(x, v_-(x)\right) & \text{if } s < v_-(x), \\ \lambda\left|s\right|^{q-2}s - f\left(x,s\right) & \text{if } v_-(x) \le s \le u_+(x), \\ \lambda u_+(x)^{q-1} - f\left(x, u_+(x)\right) & \text{if } u_+(x) < s. \end{cases} \tag{3.22}$$

Of course, $h : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function. In addition, we need the positive and negative truncations of $h_\lambda(x,\cdot)$, namely the Carathéodory functions $h_\lambda^\pm(x,s) = h_\lambda\left(x, \pm s^\pm\right)$. Let $H_\lambda\left(x,s\right) = \int_0^s h_\lambda\left(x,t\right)dt$ and $H_\lambda^\pm\left(x,s\right) = \int_0^s h_\lambda^\pm\left(x,t\right)dt$. We consider the C^1-functionals $\beta_\lambda, \beta_\lambda^\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\beta_\lambda(u) = \int_\Omega G\left(\nabla u\right)dx - \int_\Omega H_\lambda(x,u)dx,$$
$$\beta_\lambda^\pm(u) = \int_\Omega G(\nabla u)dx - \int_\Omega H_\lambda^\pm(x,u)dx.$$

Claim 1:

$$K_{\beta_\lambda} \subseteq [v_-, u_+] = \left\{u \in W_0^{1,p}(\Omega) : v_-(x) \le u(x) \le u_+(x) \text{ a.e. in } \Omega\right\},$$
$$K_{\beta_\lambda^+} = \left\{0, u_+\right\}, \quad K_{\beta_\lambda^-} = \left\{v_-, 0\right\}.$$

Let $u \in K_{\beta_\lambda}$. Then

$$A(u) = N_{h_\lambda}(u). \tag{3.23}$$

Taking $(u-u_+)^+ \in W_0^{1,p}(\Omega)$ as test function in (3.23) we derive thanks to (3.22)

$$\begin{aligned} \left\langle A(u), (u-u_+)^+\right\rangle &= \int_\Omega h_\lambda(x,u)\left(u-u_+\right)^+ dx \\ &= \int_\Omega \left[\lambda u_+^{q-1} - f(x,u_+)\right]\left(u-u_+\right)^+ dx \\ &= \left\langle A\left(u_+\right), \left(u-u_+\right)^+\right\rangle, \end{aligned}$$

hence,

$$\int_{\{u>u_+\}} \left(a(\nabla u) - a(\nabla u_+), \nabla u - \nabla u_+\right)_{\mathbb{R}^N} dx = 0.$$

This gives $|\{u > u_+\}|_N = 0$ (see Lemma 2.7(a)), thus $u \leq u_+$.

Similarly, acting on (3.23) with $(v_- - u)^+ \in W_0^{1,p}(\Omega)$ we obtain $v_- \leq u$. Hence

$$K_{\beta_\lambda} \subseteq [v_-, u_+].$$

In the same way, we show that

$$K_{\beta_\lambda^+} \subseteq [0, u_+] = \left\{u \in W_0^{1,p}(\Omega) : 0 \leq u(x) \leq u_+(x) \text{ a.e. in } \Omega\right\},$$
$$K_{\beta_\lambda^-} \subseteq [v_-, 0] = \left\{u \in W_0^{1,p}(\Omega) : v_-(x) \leq u(x) \leq 0 \text{ a.e. in } \Omega\right\}.$$

Since u_+ and v_- are the extremal constant sign solutions of $(P)_\lambda$ and due to the fact that the positive and negative solutions of $(P)_\lambda$ are exactly the critical points of β_λ^+ and β_λ^-, respectively, we infer that

$$K_{\beta_\lambda^+} = \{0, u_+\} \quad \text{and} \quad K_{\beta_\lambda^-} = \{v_-, 0\}.$$

This proves Claim 1.

Claim 2: $u_+ \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $v_- \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ are local minimizers of β_λ.

Clearly, the functional β_λ^+ is coercive and sequentially weakly lower semicontinuous. Then we find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\beta_\lambda^+(\hat{u}) = \inf\left\{\beta_\lambda^+(u) : u \in W_0^{1,p}(\Omega)\right\}.$$

As in the proof of Proposition 3.4, we can show that $\beta_\lambda^+(\hat{u}) < 0 = \beta_\lambda^+(0)$, hence $\hat{u} \neq 0$. Therefore $\hat{u} = u_+ \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ (see Claim 1). Note that $\beta_\lambda\big|_{C_0^1(\overline{\Omega})_+} = \beta_\lambda^+\big|_{C_0^1(\overline{\Omega})_+}$. Then, $u_+ \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is a local $C_0^1(\overline{\Omega})$-minimizer of β_λ and by virtue of Proposition 2.10 u_+ is a local $W_0^{1,p}(\Omega)$-minimizer of β_λ. Similarly, using β_λ^- instead of β_λ^+, we obtain the assertion for $v_- \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. This proves Claim 2.

Without loss of generality, we may assume that $\beta_\lambda(v_-) \leq \beta_\lambda(u_+)$ (the analysis is similar if the opposite inequality holds). Furthermore, we may assume that u_+ is an isolated element of K_{β_λ} (otherwise we have a whole sequence of distinct nontrivial solutions of $(P)_\lambda$). Then, there exists $\rho \in (0, 1)$ such that $\|v_- - u_+\|_{W_0^{1,p}(\Omega)} > \rho$ and

$$\beta_\lambda(v_-) \leq \beta_\lambda(u_+) < \inf\left[\beta_\lambda(u) : \|u - u_+\|_{W_0^{1,p}(\Omega)} = \rho\right] = \eta_\rho^\lambda \tag{3.24}$$

(see, for example, Aizicovici-Papageorgiou-Staicu [1, Proof of Proposition 29]). Since β_λ is coercive it satisfies the PS-condition (see Proposition 2.13). This fact and (3.24) permit the application of the Mountain Pass Theorem (Theorem 2.2), which ensures the existence of an element $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_{\beta_\lambda} \subseteq [v_-, u_+] \text{ (see Claim 1) and } \eta_\rho^\lambda \leq \beta_\lambda(y_0). \tag{3.25}$$

From (3.25) it follows that y_0 is a solution of $(P)_\lambda$ (see (3.22)) and $y_0 \notin \{v_-, u_+\}$ (see also (3.25)). The nonlinear regularity theory implies that $y_0 \in C_0^1(\overline{\Omega})$. Since $y_0 \in [v_-, u_+] \setminus \{v_-, u_+\}$, the extremality of v_- and u_+ implies that y_0 is nodal provided $y_0 \neq 0$.

Thanks to Theorem 2.2 we also have that

$$\beta_\lambda(y_0) = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \beta_\lambda(\gamma(t)), \tag{3.26}$$

where $\Gamma = \left\{\gamma \in C\left([0,1], W_0^{1,p}(\Omega)\right) : \gamma(0) = v_-, \gamma(1) = u_+\right\}$. In order to show the nontriviality of y_0 it suffices to produce a path $\gamma_* \in \Gamma$ such that $\beta_\lambda(\gamma_*(t)) \neq 0$ for all $t \in [0,1]$ (see (3.26)).

By means of hypothesis H(f)$_1$(iii), given $\varepsilon > 0$, we find $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x,s)| \leq \varepsilon |s|^{q-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \leq \delta,$$

which implies that

$$F(x,s) \leq \frac{\varepsilon}{q}|s|^q \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \leq \delta. \tag{3.27}$$

Recall (see Section 2) that $\partial B_1^{L^q} = \{u \in L^q(\Omega) : \|u\|_{L^q(\Omega)} = 1\}$ and $M_q = W_0^{1,q}(\Omega) \cap \partial B_1^{L^q}$. Defining $M_q^c = M_q \cap C_0^1(\overline{\Omega})$, we endow M_q with the relative $W_0^{1,p}(\Omega)$-topology and M_q^c with the relative $C_0^1(\overline{\Omega})$-topology. Evidently, M_q^c is dense in M_q which implies that $C\left([-1,1], M_q^c\right)$ is dense in $C\left([-1,1], M_q\right)$ as well. We consider the following two sets of continuous paths

$$\hat{\Gamma}(q) = \left\{\hat{\gamma} \in C\left([-1,1], M_q\right) : \hat{\gamma}(-1) = -\hat{u}_1(q), \hat{\gamma}(1) = \hat{u}_1(q)\right\}, \text{(cf. Section 2)},$$
$$\hat{\Gamma}_c(q) = \left\{\hat{\gamma} \in C\left([-1,1], M_q^c\right) : \hat{\gamma}(-1) = -\hat{u}_1(q), \hat{\gamma}(1) = \hat{u}_1(q)\right\}.$$

Note that $\hat{\Gamma}_c(q)$ is dense in $\hat{\Gamma}(q)$. Taking into account the variational characterization of the second eigenvalue $\hat{\lambda}_2(q)$ (see Proposition 2.12), we find $\hat{\gamma} \in \hat{\Gamma}(q)$ such that

$$\max_{-1 \leq t \leq 1} \|\nabla \hat{\gamma}(t)\|_{L^q(\Omega)}^q < \hat{\lambda}_2(q) + \frac{\delta}{2}. \tag{3.28}$$

The density of $\hat{\Gamma}_c(q)$ in $\hat{\Gamma}(q)$ implies that for a given $\varepsilon > 0$ there exists $\hat{\gamma}_0 \in \hat{\Gamma}_c(q)$ such that

$$\max_{-1 \leq t \leq 1} \|\hat{\gamma}(t) - \hat{\gamma}_0(t)\|_{W_0^{1,q}(\Omega)} < \varepsilon,$$

from which we derive that

$$\max_{-1 \leq t \leq 1} \|\nabla \hat{\gamma}(t) - \nabla \hat{\gamma}_0(t)\|_{L^q(\Omega)} < \varepsilon. \tag{3.29}$$

Combining (3.28) and (3.29), there holds, for every $t \in [-1,1]$,

$$\begin{aligned}\|\nabla \hat{\gamma}_0(t)\|_{L^q(\Omega)} &\leq \|\nabla \hat{\gamma}_0(t) - \nabla \hat{\gamma}(t)\|_{L^q(\Omega)} + \|\nabla \hat{\gamma}(t)\|_{L^q(\Omega)} \\ &< \varepsilon + \left(\hat{\lambda}_2(q) + \frac{\delta}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Choosing $0 < \varepsilon < \left(\hat{\lambda}_2(q) + \delta\right)^{\frac{1}{q}} - \left(\hat{\lambda}_2(q) + \frac{\delta}{2}\right)^{\frac{1}{q}}$, we obtain

$$\|\nabla \hat{\gamma}_0(t)\|_{L^q(\Omega)} < \left(\hat{\lambda}_2(q) + \delta\right)^{\frac{1}{q}} \quad \text{for all } t \in [-1,1],$$

meaning that

$$\max_{-1 \leq t \leq 1} \|\nabla \hat{\gamma}_0(t)\|_{L^q(\Omega)}^q < \hat{\lambda}_2(q) + \delta. \tag{3.30}$$

Since $\hat{\gamma}_0([-1,1]) \subseteq C_0^1(\overline{\Omega})$ is compact and due to $u_+ \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, $v_- \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, we can find $\eta \in (0,1)$ such that

$$|\eta u(x)| \leq \delta \tag{3.31}$$

for all $x \in \overline{\Omega}$, all $u \in \hat{\gamma}_0([-1,1])$, and $\eta u \in [v_-, u_+]$ for all $u \in \hat{\gamma}_0([-1,1])$.

First let us suppose that $q < p$. Then, thanks to Corollary 2.8, $\|u\|_{L^q(\Omega)} = 1$, (3.27), (3.30), (3.31), and $u \in \hat{\gamma}_0([-1,1])$, it follows

$$\begin{aligned}\beta_\lambda(\eta u) &= \int_\Omega G(\nabla(\eta u))dx - \int_\Omega H_\lambda(x,\eta u)dx \\ &\le c_5\left(\eta^q\|\nabla u\|^q_{L^q(\Omega)} + \eta^p\|\nabla u\|^p_{L^p(\Omega)}\right) - \frac{\lambda\eta^q}{q} + \int_\Omega F(x,\eta u)dx \\ &\le c_5\left(\eta^q\|\nabla u\|^q_{L^q(\Omega)} + \eta^p\|\nabla u\|^p_{L^p(\Omega)}\right) - \frac{\lambda\eta^q}{q} + \frac{\varepsilon}{q}\eta^q \\ &\le \eta^q\left[c_5\left(\hat{\lambda}_2(q)+\delta\right) - \frac{\lambda-\varepsilon}{q}\right] + c_5\eta^p\|\nabla u\|^p_{L^p(\Omega)}.\end{aligned} \tag{3.32}$$

Since $\hat{\gamma}_0([-1,1]) \subseteq C^1_0(\overline{\Omega})$ is compact, there exists $\varepsilon^* > 0$ such that

$$\|\nabla u\|^p_{L^p(\Omega)} \le \varepsilon^* \quad \text{for all } u \in \hat{\gamma}_0([-1,1]).$$

We choose $\varepsilon > 0$ and $\delta \in (0,\varepsilon)$ such that $qc_5\delta + \varepsilon < \lambda - qc_5\hat{\lambda}_2(q)$. Then, due to (3.32), it holds

$$\beta_\lambda(\eta u) \le -\eta^q\hat{\varepsilon}^* + \eta^p\varepsilon^* \quad \text{for some } \hat{\varepsilon}^* > 0.$$

Because $q < p$, by choosing $\eta \in (0,1)$ even smaller if necessary, we get

$$\beta_\lambda(\eta u) < 0 \quad \text{for all } u \in \hat{\gamma}_0([-1,1]).$$

If $q = p$, the above argument works for $\lambda > 2qc_5\hat{\lambda}_2(q)$.

Setting $\gamma_0 = \eta\hat{\gamma}_0$, we see that γ_0 is a continuous path in $W^{1,p}_0(\Omega)$ $\left(\text{since } \gamma_0([-1,1]) \subseteq C^1_0(\overline{\Omega})\right)$, which connects $-\eta\hat{u}_1(q)$ and $\eta\hat{u}_1(q)$ satisfying

$$\beta_\lambda\big|_{\gamma_0} < 0. \tag{3.33}$$

Recalling that $K_{\beta^+_\lambda} = \{0, u_+\}$ (see Claim 1), we have

$$\beta^+_\lambda(u_+) = \inf_{u \in W^{1,p}_0(\Omega)} \beta^+_\lambda(u) < 0 = \beta^+_\lambda(0) \tag{3.34}$$

(cf. the proof of Claim 2). Invoking the Second Deformation Lemma (see Lemma 2.3) with $a = \beta^+_\lambda(u_+) < 0 = \beta^+_\lambda(0) = b$, we find a homotopy $h : [0,1] \times \left(\left(\beta^+_\lambda\right)^0 \setminus \{0\}\right) \to \left(\beta^+_\lambda\right)^0$ such that

$$h\left(1, \left(\beta^+_\lambda\right)^0 \setminus \{0\}\right) = \{u_+\} \tag{3.35}$$

(see (3.34) and Claim 1) and

$$\beta^+_\lambda(h(t,u)) \le \beta^+_\lambda(u) \quad \text{for all } t \in [0,1] \text{ and all } u \in \left(\beta^+_\lambda\right)^0 \setminus \{0\}. \tag{3.36}$$

We set $\gamma_+(t) = (h(t,\eta\hat{u}_1(q)))^+$ for all $t \in [0,1]$ (see (3.33)). Obviously, γ_+ is a continuous path in $W^{1,p}_0(\Omega)$ and we have

$$\begin{aligned}\gamma_+(0) &= (h(0,\eta\hat{u}_1(q)))^+ = \eta\hat{u}_1(q) \\ \gamma_+(1) &= (h(1,\eta\hat{u}_1(q))^+ = u_+,\end{aligned}$$

due to (3.35). Furthermore, (3.36) and (3.33) imply that

$$\beta_\lambda^+(\gamma_+(t)) \le \beta_\lambda^+(\eta\hat{u}_1(q)) < 0 \quad \text{for all } t \in [0,1].$$

Hence, $\beta_\lambda^+\big|_{\gamma_+} < 0$. If $W_+^p = \{u \in W_0^{1,p}(\Omega) : u(x) \ge 0 \text{ a.e. in } \Omega\}$, then

$$\beta_\lambda\big|_{W_+^p} = \beta_\lambda^+\big|_{W_+^p} \quad \text{and} \quad \operatorname{im}\gamma_+ \subseteq W_+^p,$$

which yields

$$\beta_\lambda\big|_{\gamma_+} < 0. \tag{3.37}$$

In a similar fashion, we produce another continuous path γ_- in $W_0^{1,p}(\Omega)$ which connects $-\eta\hat{u}_1(q)$ and v_-. Furthermore, we have again

$$\beta_\lambda\big|_{\gamma_-} < 0. \tag{3.38}$$

We concatenate γ_-, γ_0, and γ_+ to obtain a continuous path $\gamma_* \in \Gamma$ such that

$$\beta_\lambda\big|_{\gamma_*} < 0,$$

due to (3.33), (3.37), and (3.38). This implies that $y_0 \ne 0$, so $y_0 \in C_0^1(\overline{\Omega})$ is a nodal solution of $(P)_\lambda$. □

Now, we can state the following multiplicity result for problem $(P)_\lambda$.

Theorem 3.8 *If hypotheses H(a)$_1$ and H(f)$_1$ hold and assume*

$$\lambda > \begin{cases} qc_6\hat{\lambda}_2(q) & \text{if } q < p, \\ 2qc_6\hat{\lambda}_2(q) & \text{if } q = p, \end{cases}$$

then problem $(P)_\lambda$ has at least three nontrivial solutions, namely $u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, $v_0 \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and a nodal solution $y_0 \in [v_0, u_0] \cap C_0^1(\overline{\Omega})$. Furthermore, $(P)_\lambda$ has a smallest nontrivial positive solution $u_+ \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and a greatest nontrivial negative solution $v_- \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Remark 3.9 Note that if $q = p$ and $a(\xi) = \|\xi\|^{p-2}\xi$ for all $\xi \in \mathbb{R}^N$, then $c_6 = \frac{1}{2q}$ and consequently $2qc_6\hat{\lambda}_2(q) = \hat{\lambda}_2(q)$. Therefore, Theorem 3.8 recovers the multiplicity results of [2], [3], [33], [34] (where $p = 2$) and [5], [8], [29] (where $p > 1$). In fact, even in this special case, our result is more general than those in the aforementioned works. Indeed, in the semilinear works (i.e., $p = 2$), the perturbation $f(x, s) = f(s)$ is either C^1 (see [2], [3], [33]) or Lipschitz continuous (see [34]). In the quasilinear works (i.e., $p > 1$, see [5], [29]), the perturbation $f(x, s)$ is assumed to have strictly subcritical growth in the variable $s \in \mathbb{R}$. In contrast here no growth restriction is imposed on $f(x, \cdot)$ (see hypothesis H(f)$_1$(i)).

4 The $(p, 2)$-equation

In this section we consider the special case $a(\xi) = \|\xi\|^{p-2}\xi + \xi$ for all $\xi \in \mathbb{R}^N$ with $2 < p < \infty$ and $q = 2$. Then, problem $(P)_\lambda$ becomes

$$\begin{aligned} -\Delta_p u - \Delta u &= \lambda u - f(x,u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{Q}_\lambda$$

Applying Morse theory (critical groups), we are going to show that for all $\lambda > 0$ sufficiently large not being in the spectrum of $(-\Delta, H_0^1(\Omega))$, problem $(Q)_\lambda$ has at least four nontrivial smooth solutions whereby two of them have constant sign and the other ones have changing sign. To do so, we need to strengthen our hypotheses on the perturbation $f : \Omega \times \mathbb{R} \to \mathbb{R}$ in the following way.

H(f)$_2$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that $f(x, \cdot) \in C^1(\mathbb{R})$ for a.a. $x \in \Omega$ and

(i) $\left|f_s'(x, s)\right| \leq c|s|^{r-2}$ for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, with $c > 0$, and $p < r < p^*$;

(ii) $\lim\limits_{s \to \pm\infty} \dfrac{f(x, s)}{|s|^{p-2}s} = +\infty$ uniformly for a.a. $x \in \Omega$;

(iii) $f_s'(x, 0) = \lim\limits_{s \to 0} \dfrac{f(x, s)}{s} = 0$ uniformly for a.a. $x \in \Omega$.

Remark 4.1 These hypotheses are similar to those in H(f)$_1$. However, we now require that $f(x, \cdot)$ is differentiable for a.a. $x \in \Omega$ and exhibits strictly subcritical growth. Note that we can find $\varepsilon^* > 0$ such that $s \mapsto \lambda s - f(x, s) + \varepsilon^*|s|^{r-2}s$ is nondecreasing on $\mathbb{R}$ for a.a. $x \in \Omega$.

We have the following multiplicity theorem for problem $(Q)_\lambda$.

Theorem 4.2 *Let hypotheses H(f)$_2$ be satisfied and let $\lambda > \hat{\lambda}_2(2), \lambda \notin \hat{\sigma}(2)$, then problem $(Q)_\lambda$ has at least four nontrivial solutions whereby two of them have constant sign, namely $u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), v_0 \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, and two of them have changing sign, namely $y_0, \hat{y} \in C_0^1(\overline{\Omega})$.*

Proof. Taking into account Theorem 3.8 we already know the existence of three nontrivial solutions

$$u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), v_0 \in -\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), \text{ and } y_0 \in [v_0, u_0] \cap C_0^1(\overline{\Omega}).$$

Moreover, we may assume that u_0, v_0 are extremal constant sign solutions. In the present case we have $a(\xi) = \|\xi\|^{p-2}\xi + \xi$ for all $\xi \in \mathbb{R}^N$. Hence,

$$\nabla a(\xi) = \|\xi\|^{p-2}\left(I + (p-2)\frac{\xi \otimes \xi}{\|\xi\|^2}\right) + I \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}.$$

It follows that

$$(\nabla a(\xi)y, y)_{\mathbb{R}^N} \geq \|y\|^2 \quad \text{for all } \xi, y \in \mathbb{R}^N.$$

Invoking the tangency principle of Pucci-Serrin [31, Theorem 2.5.2], we have

$$v_0(x) < y_0(x) < u_0(x) \quad \text{for all } x \in \Omega. \tag{4.1}$$

As already observed in Remark 4.1, there exists $\varepsilon^* > 0$ such that

$$s \mapsto \lambda s - f(x, s) + \varepsilon^*|s|^{r-2}s \text{ is nondecreasing on } \mathbb{R} \text{ for a.a. } x \in \Omega. \tag{4.2}$$

Applying (4.1) and (4.2) we derive

$$\begin{aligned} -\Delta_p u_0(x) - \Delta u_0(x) + \varepsilon^* u_0(x)^{r-1} &= \lambda u_0(x) - f(x, u_0(x)) + \varepsilon^* u_0(x)^{r-1} \\ &\geq \lambda y_0(x) - f(x, y_0(x)) + \varepsilon^*|y_0(x)|^{r-2}y_0(x) \\ &= -\Delta_p y_0(x) - \Delta y_0(x) + \varepsilon^*|y_0(x)|^{r-2}y_0(x) \end{aligned}$$

almost everywhere in Ω. Since $y_0, u_0 \in C^1_0(\overline{\Omega})$ and (4.1) holds, we infer from Proposition 2.5 that

$$y_0 \in \operatorname*{int}_{C^1_0(\overline{\Omega})} [v_0, u_0]. \tag{4.3}$$

The energy functional of problem $(Q)_\lambda$, namely $\varphi_\lambda : W^{1,p}_0(\Omega) \to \mathbb{R}$, is defined by

$$\varphi_\lambda(u) = \frac{1}{p}\|\nabla u\|^p_{L^p(\Omega)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} + \int_\Omega F(x,u)dx.$$

Evidently, $\varphi_\lambda \in C^2(W^{1,p}_0(\Omega))$. Let $\beta_\lambda : W^{1,p}_0(\Omega) \to \mathbb{R}$ be the functional obtained by truncating the reaction of $(Q)_\lambda$ at $\{v_0(x), u_0(x)\}$ as in the proof of Proposition 3.7. We have $\beta_\lambda \in C^{2-0}(W^{1,p}_0(\Omega))$ and

$$\beta_\lambda\big|_{[v_0,u_0]} = \varphi_\lambda\big|_{[v_0,u_0]}$$

(see (3.22)). Both functionals are coercive, consequently they satisfy the PS-condition (see Proposition 2.13). We consider the homotopy

$$h(t,u) = t\beta_\lambda(u) + (1-t)\varphi_\lambda(u) \quad \text{for all } t \in [0,1] \text{ and all } u \in W^{1,p}_0(\Omega).$$

Suppose that we can find $(t_n)_{n\geq 1} \subseteq [0,1]$ and $(u_n)_{n\geq 1} \subseteq W^{1,p}_0(\Omega)$ such that

$$t_n \to t, u_n \to y_0 \text{ in } W^{1,p}_0(\Omega) \quad \text{and} \quad h'_u(t_n, u_n) = 0 \quad \text{for all } n \geq 1. \tag{4.4}$$

Then, we have

$$\Delta_p u_n + \Delta u_n = t_n N_{h_\lambda}(u_n) + (1-t)N_f(u_n)$$

meaning that

$$\begin{aligned} -\Delta_p u_n - \Delta u_n &= t_n h_\lambda(x,u_n) + (1-t)f(x,u_n) && \text{in } \Omega, \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.5}$$

for all $n \geq 1$. From Ladyzhenskaya-Ural′tseva [25, p. 286] we know that the solutions of (4.5) are essentially bounded, i.e., there exists $M_2 > 0$ such that

$$\|u_n\|_{L^\infty(\Omega)} \leq M_2 \quad \text{for all } n \geq 1.$$

Then, the regularity results of Lieberman [26, p. 320] imply the existence of $\gamma \in (0,1)$ and $M_3 > 0$ such that

$$u_n \in C^{1,\gamma}_0(\overline{\Omega}) \text{ and } \|u_n\|_{C^{1,\gamma}_0(\overline{\Omega})} \leq M_3 \quad \text{for all } n \geq 1.$$

Since $C^{1,\gamma}_0(\overline{\Omega})$ is compactly embedded in $C^1_0(\overline{\Omega})$, it follows from (4.4) that

$$u_n \to y_0 \text{ in } C^1_0(\overline{\Omega}).$$

Due to (4.3) we conclude that

$$u_n \in [v_0, u_0] \setminus \{0\} \quad \text{for all } n \geq n_0.$$

Hence, the sequence $(u_n)_{n\geq n_0}$ contains only nodal solutions of $(Q)_\lambda$ and from there we are done.

Accordingly, we may assume that there exists $\rho \in (0,1)$ small such that

$$\overline{B}_\rho(y_0) \cap K_{h(t,0)} = \{y_0\} \quad \text{for all } t \in [0,1]$$

with

$$\overline{B}_\rho(y_0) = \left\{u \in W_0^{1,p}(\Omega) : \|u - y_0\| \leq \rho\right\}.$$

Invoking the homotopy invariance property of critical groups, we have

$$C_k(\varphi_\lambda, y_0) = C_k(\beta_\lambda, y_0) \quad \text{for all } k \geq 0. \tag{4.6}$$

Owing to the proof of Proposition 3.7 we know that y_0 is a critical point of β_λ (i.e., $y_0 \in K_{\beta_\lambda}$) of mountain pass type. Therefore, due to Chang [10],

$$C_1(\beta_\lambda, y_0) \neq 0$$

which combined with (4.6) yields

$$C_1(\varphi_\lambda, y_0) \neq 0. \tag{4.7}$$

Since $\varphi_\lambda \in C^2(W_0^{1,p}(\Omega))$, from (4.7) and Papageorgiou-Smyrlis [30], we have

$$C_k(\varphi_\lambda, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0$$

which thanks to (4.6) gives

$$C_k(\beta_\lambda, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.8}$$

Because of hypothesis $\lambda > \hat{\lambda}_2(2), \lambda \notin \hat{\sigma}(2)$, there is a number $m \geq 2$ such that $\lambda \in \left(\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)\right)$. Let $\varepsilon\left(0, \hat{\lambda}_m(2)\right)$. By virtue of hypothesis $\mathrm{H(f)}_2$(iii), we find $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x,s)| \leq \varepsilon|s| \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \leq \delta.$$

It follows that

$$F(x,s) \leq \frac{\varepsilon}{2}s^2 \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \leq \delta. \tag{4.9}$$

Let $\Theta_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^2-functional defined by

$$\Theta_\lambda(u) = \frac{1}{p}\|\nabla u\|_{L^p(\Omega)}^p + \frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2 - \frac{\lambda}{2}\|u\|_{L^2(\Omega)}^2.$$

Apparently, Θ is coercive, hence, it satisfies the PS-condition. Then, from (4.9) and Chang [10, Corollary 5.1.25 (p. 336)], we verify that

$$C_k(\varphi_\lambda, 0) = C_k(\Theta_\lambda, 0) \quad \text{for all } k \geq 0,$$

which implies (see (4.6)) that

$$C_k(\beta_\lambda, 0) = C_k(\Theta_\lambda, 0) \quad \text{for all } k \geq 0. \tag{4.10}$$

By reason of $\lambda \in \left(\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)\right)$ and due to Cingolani-Vannella [13, Theorem 1.1] one has

$$C_k(\Theta_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geq 0,$$

where $d_m = \dim \bigoplus_{i=1}^{m} E\left(\hat{\lambda}_i(2)\right) \geq 2$. Owing to (4.10) we derive that

$$C_k(\beta_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.11}$$

From the proof of Proposition 3.7 (see Claim 2), we know that u_0, v_0 are both local minimizers of β_λ. Hence

$$C_k(\beta_\lambda, u_0) = C_k(\beta_\lambda, v_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.12}$$

Recalling that β_λ is coercive, it follows that

$$C_k(\beta_\lambda, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.13}$$

Suppose that $K_{\beta_\lambda} = \{0, u_0, v_0, y_0\}$. Then, from (4.8), (4.11), (4.12), (4.13) and the Morse relation (2.6) with $t = -1$, we have

$$2(-1)^0 + (-1)^1 + (-1)^{d_m} = (-1)^0,$$

which gives $(-1)^{d_m} = 0$, a contradiction. Therefore, we can find $\hat{y} \in K_{\beta_\lambda}, \hat{y} \notin \{0, u_0, v_0, y_0\}$. Since $K_{\beta_\lambda} \subseteq [v_0, u_0]$ (see the proof of Claim 1 in the proof of Proposition 3.7), one gets $\hat{y} \in [v_0, u_0] \setminus \{0\}$. The extremal property of u_0 and v_0 implies that $\hat{y}$ has changing sign. The nonlinear regularity theory (see [25], [26]) yields $\hat{y} \in C_0^1(\overline{\Omega})$. That finishes the proof.

□

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Parametric nonlinear nonhomogeneous Neumann equations involving a nonhomogeneous differential operator

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Abstract This work is concerned with the existence of solutions to parametric elliptic equations driven by a nonhomogeneous differential operator with a nonhomogeneous Neumann boundary condition. The assumptions on the operator involve the p-Laplacian, the (p, q)-Laplacian, and the generalized p-mean curvature differential operator. Based on variational tools combined with truncation and comparison techniques we prove the existence of at least three nontrivial solutions provided the parameter is sufficiently large whereby the first solution is positive, the second one is negative and the third one has changing sign (nodal).

Keywords Constant sign and nodal solutions · Nonlinear regularity · Critical point of mountain pass type · Extremal solutions · Local minimizers · Maximum principle

Mathematics Subject Classification (2010) 35J20 · 35J60 · 35J92 · 58E05

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $1 < q \leq p$. We study the following nonlinear nonhomogeneous Neumann problem

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$$\begin{aligned} -\operatorname{div} a(\nabla u) &= -\chi |u|^{p-2}u - f(x,u) \qquad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n_a} &= \lambda |u|^{q-2}u - h(x,u) \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $\partial u/\partial n_a$ denotes the conormal derivative with respect to the mapping $a : \mathbb{R}^N \to \mathbb{R}^N$ which is supposed to be continuous and strictly monotone with $(p-1)$-growth. The nonlinearities $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are assumed to be Carathéodory functions being $(p-1)$-superlinear near $\pm\infty$ and bounded on bounded sets while χ, λ are real parameters to be specified. The aim of this paper is to establish the existence of at least three nontrivial solutions provided $\lambda > 0$ is sufficiently large depending on the first two eigenvalues of the negative q-Laplacian $-\Delta_q$ with Steklov boundary condition. In addition we give complete sign information of the solutions obtained, that is, the first solution is positive, the second one is negative and finally, the third one has changing sign.

Such results are known for quasilinear elliptic equations involving the p-Laplacian and were obtained by a number of authors in the last years with different methods. Without guarantee of completeness we refer to the papers of Abreu-Marcos do Ó-Medeiros [1], Fernández Bonder-Rossi [6], Fernández Bonder [7,8], Li-Li [14], Liu-Zheng [16], Martínez-Rossi [18], Winkert [23,25], Zhao-Zhao [27], and the references therein. To the best of our knowledge, the results presented here are the first one concerning multiplicity of solutions for equations involving a nonhomogeneous operator with nonhomogeneous Neumann boundary condition.

Our paper extends the results of Winkert [23] in different ways. On the one hand we can replace the p-Laplacian used in [23] by a more general nonhomogeneous operator and on the other hand we can drop a hypothesis on the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ required in [23], namely:

(H) There exists a number $\delta_f > 0$ such that $\dfrac{f(x,s)}{|s|^{p-2}s} \geq 0$ for all $0 < |s| \leq \delta_f$ and for a.a. $x \in \Omega$.

Assumption (H) implies that the function f must change sign near zero. Now, we do not need this condition on f. It is also worth pointing out that we do not need differentiability, polynomial growth or some integral conditions on the mappings f and h. Our approach is based on variational methods coupled with truncation and comparison techniques.

2 Preliminaries and hypotheses

In this section we recall some basic facts about critical point theory which will be needed in Sect. 3. For this purpose, let X be a Banach space with norm $\|\cdot\|_X$ and denote by X^* its dual space equipped with the dual norm $\|\cdot\|_{X^*}$, that is

$$\|\xi\|_{X^*} = \sup\left\{\langle \xi, v\rangle_{(X^*,X)} : v \in X, \|v\|_X \leq 1\right\},$$

where $\langle\cdot,\cdot\rangle_{(X^*,X)}$ stands for the duality paring of (X^*, X).

Definition 2.1 The functional $\varphi \in C^1(X)$ fulfills the Palais-Smale condition at the level $c \in \mathbb{R}$ (the PS_c-condition for short) if every sequence $(u_n)_{n\geq 1} \subseteq X$ satisfying $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ in X^*, admits a strongly convergent subsequence. We say that φ satisfies the Palais-Smale condition (the PS-condition for short) if it satisfies the PS_c-condition for every $c \in \mathbb{R}$.

This compactness-type condition on φ leads to a deformation theorem which is the main ingredient in the minimax theory of the critical values of φ. A basic result in that theory is the so-called mountain pass theorem.

Theorem 2.2 *If $\varphi \in C^1(X)$, $u_1, u_2 \in X$, $\|u_2 - u_1\|_X > \rho > 0$,*

$$\max\{\varphi(u_1), \varphi(u_2)\} < \inf\{\varphi(u) : \|u - u_1\|_X = \rho\} =: m_\rho$$

and φ satisfies the PS_c-condition where

$$c = \inf_{\gamma \in \Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$$

with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_1, \gamma(1) = u_2\}$, then $c \geq m_\rho$ and c is a critical value of φ.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\begin{aligned}
&\varphi^c = \{u \in X : \varphi(u) \leq c\} && \text{(the sublevel set of } \varphi \text{ at } c),\\
&K_\varphi = \{u \in X : \varphi'(u) = 0\} && \text{(the critical set of } \varphi),\\
&K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\} && \text{(the critical set of } \varphi \text{ at the level } c).
\end{aligned}$$

The following result is the so-called second deformation theorem (see, for example, Gasiński and Papageorgiou [11, p. 628]).

Theorem 2.3 *If $\varphi \in C^1(X)$, $a \in \mathbb{R}$, $a < b \leq +\infty$, φ satisfies the PS_c-condition for every $c \in [a, b)$, φ has no critical values in (a, b) and $\varphi^{-1}(a)$ contains at most a finite number of critical points of φ, then there exists a continuous map $\hat{h} : [0,1] \times \left(\varphi^b \backslash K_\varphi^b\right) \to \varphi^b$ such that*

(a) $\hat{h}(0, u) = u$ for all $u \in \varphi^b \backslash K_\varphi^b$;
(b) $\hat{h}(1, \varphi^b \backslash K_\varphi^b) \subseteq \varphi^a$;
(c) $\hat{h}(t, u) = u$ for all $(t, u) \in [0,1] \times \varphi^a$;
(d) $\varphi(\hat{h}(t, u)) \leq \varphi(\hat{h}(s, u))$ for all $t, s \in [0,1]$, $s \leq t$, and all $u \in \varphi^b \backslash K_\varphi^b$.

By $L^p(\Omega)$ $\left(\text{or } L^p\left(\Omega; \mathbb{R}^N\right)\right)$, $L^p(\partial\Omega)$, and $W^{1,p}(\Omega)$ we denote the usual Lebesgue and Sobolev spaces with their norms $\|\cdot\|_{p,\Omega}$, $\|\cdot\|_{p,\partial\Omega}$, respectively, $\|\cdot\|_{1,p}$, which is given by

$$\|u\|_{1,p} = \left(\|\nabla u\|_{p,\Omega}^p + \|u\|_{p,\Omega}^p\right)^{\frac{1}{p}} \quad \text{for all} \quad u \in W^{1,p}(\Omega).$$

The norm of $\mathbb{R}^N$ is denoted by $\|\cdot\|_{\mathbb{R}^N}$ and $(\cdot,\cdot)_{\mathbb{R}^N}$ stands for the inner product in $\mathbb{R}^N$. In addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the ordered Banach space $C^1(\overline{\Omega})$ with norm $\|\cdot\|_{C^1(\overline{\Omega})}$ and its positive cone

$$C^1(\overline{\Omega})_+ = \left\{u \in C^1(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\},$$

which has a nonempty interior given by

$$\text{int}\left(C^1(\overline{\Omega})_+\right) = \left\{u \in C^1(\overline{\Omega})_+ : u(x) > 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

Now let $\omega \in C^1(0,+\infty)$ be a function satisfying

$$0 < c_0 \leq \frac{t\omega'(t)}{\omega(t)} \leq c_1 \quad \text{and} \quad c_2 t^{p-1} \leq \omega(t) \leq c_3(1+t^{p-1})$$

for all $t > 0$ and with some constants $c_0, c_1, c_2, c_3 > 0$. The hypotheses on $a : \mathbb{R}^N \to \mathbb{R}^N$ read as follows.

H(a): $a(\xi) = a_0\left(\|\xi\|_{\mathbb{R}^N}\right)\xi$ for all $\xi \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,\infty)$, $t \mapsto ta_0(t)$ is strictly increasing, $\lim_{t\to 0^+} ta_0(t) = 0$, and $\lim\limits_{t\to 0^+} \dfrac{ta_0'(t)}{a_0(t)} > -1$;

(ii) $\|\nabla a(\xi)\|_{\mathbb{R}^N} \leq c_4 \dfrac{\omega\left(\|\xi\|_{\mathbb{R}^N}\right)}{\|\xi\|_{\mathbb{R}^N}}$ for all $\xi \in \mathbb{R}^N\backslash\{0\}$ and some $c_4 > 0$;

(iii) $(\nabla a(\xi)y, y)_{\mathbb{R}^N} \geq \dfrac{\omega\left(\|\xi\|_{\mathbb{R}^N}\right)}{\|\xi\|_{\mathbb{R}^N}}\|y\|^2_{\mathbb{R}^N}$ for all $\xi \in \mathbb{R}^N\backslash\{0\}$ and all $y \in \mathbb{R}^N$.

It is easy to see that condition H(a)(i) implies that $a \in C^1\left(\mathbb{R}^N\backslash\{0\}, \mathbb{R}^N\right) \cap C\left(\mathbb{R}^N, \mathbb{R}^N\right)$ and so, the assumptions in hypotheses H(a)(ii), (iii) are reasonable.

Let $G_0(t) = \int_0^t sa_0(s)ds$ and let $G(\xi) = G_0\left(\|\xi\|_{\mathbb{R}^N}\right)$ for all $\xi \in \mathbb{R}^N$. Then

$$\nabla G(\xi) = G_0'\left(\|\xi\|_{\mathbb{R}^N}\right)\frac{\xi}{\|\xi\|_{\mathbb{R}^N}} = a_0\left(\|\xi\|_{\mathbb{R}^N}\right)\xi = a(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^N\backslash\{0\},$$

which means that $G(\cdot)$ is the primitive of $a(\cdot)$. Obviously, $G(\cdot)$ is convex and since $G(0) = 0$ we have the estimate

$$G(\xi) \leq (a(\xi), \xi)_{\mathbb{R}^N} \quad \text{for all} \quad \xi \in \mathbb{R}^N. \tag{2.1}$$

The following lemma gives some basic properties of the mapping $a : \mathbb{R}^N \to \mathbb{R}^N$.

Lemma 2.4 *Under the hypotheses H(a) there holds*

(i) $\xi \to a(\xi)$ *is maximal monotone and strictly monotone;*

(ii) $\|a(\xi)\|_{\mathbb{R}^N} \leq c_5\left(1 + \|\xi\|^{p-1}_{\mathbb{R}^N}\right)$ *for all* $\xi \in \mathbb{R}^N$ *and some* $c_5 > 0$*;*

(iii) $(a(\xi),\xi)_{\mathbb{R}^N} \geq \frac{c_2}{p-1}\|\xi\|^p_{\mathbb{R}^N}$ *for all* $\xi \in \mathbb{R}^N$.

Taking into account Lemma 2.4 combined with (2.1) we infer the following growth estimates for the primitive $G(\cdot)$.

Corollary 2.5 *If hypotheses H(a) hold, then*

$$\frac{c_2}{p(p-1)}\|\xi\|^p_{\mathbb{R}^N} \leq G(\xi) \leq c_6\left(1+\|\xi\|^p_{\mathbb{R}^N}\right) \quad \text{for all} \quad \xi \in \mathbb{R}^N \text{ and some } \quad c_6 > 0.$$

It should be mentioned that the operator $a : \mathbb{R}^N \to \mathbb{R}^N$ defined through hypotheses H(a) contains several interesting differential operators as special cases.

(i) Let $1 < p < \infty$ and let $a(\xi) = \|\xi\|^{p-2}_{\mathbb{R}^N}\xi$ with $1 < p < \infty$. Then $a(\cdot)$ represents the well-known p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(\|\nabla u\|^{p-2}_{\mathbb{R}^N}\nabla u\right) \quad \text{for all} \quad u \in W^{1,p}(\Omega).$$

The corresponding potential is given by $G(\xi) = \frac{1}{p}\|\xi\|^p_{\mathbb{R}^N}$ for all $\xi \in \mathbb{R}^N$.

(ii) Let $1 < q < p$ and let $a(\xi) = \|\xi\|^{p-2}_{\mathbb{R}^N}\xi + \|\xi\|^{q-2}_{\mathbb{R}^N}\xi$. Then $a(\cdot)$ becomes the (p,q)-differential operator defined by

$$\Delta_p u + \Delta_q u = \operatorname{div}\left(\|\nabla u\|^{p-2}_{\mathbb{R}^N}\nabla u\right) + \operatorname{div}\left(\|\nabla u\|^{q-2}_{\mathbb{R}^N}\nabla u\right)$$

for all $u \in W^{1,p}(\Omega)$. The associated potential is $G(\xi) = \frac{1}{p}\|\xi\|^p_{\mathbb{R}^N} + \frac{1}{q}\|\xi\|^q_{\mathbb{R}^N}$ for all $\xi \in \mathbb{R}^N$.

(iii) Let $1 < p < \infty$ and let $a(\xi) = \left(1+\|\xi\|^2_{\mathbb{R}^N}\right)^{\frac{p-2}{2}}\xi$. In this case $a(\cdot)$ represents the generalized p-mean curvature differential operator which is defined by

$$\operatorname{div}\left[\left(1+\|\nabla u\|^2_{\mathbb{R}^N}\right)^{\frac{p-2}{2}}\nabla u\right] \quad \text{for all} \quad u \in W^{1,p}(\Omega).$$

The potential is $G(\xi) = \frac{1}{p}\left[\left(1+\|\xi\|^2_{\mathbb{R}^N}\right)^{\frac{p}{2}} - 1\right]$ for all $\xi \in \mathbb{R}^N$.

Now, let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$, $h_0 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions satisfying the subsequent growth conditions

$$|f_0(x,s)| \leq c_{f_0}\left(1+|s|^{r_1-1}\right) \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R},$$
$$|h_0(x,s)| \leq c_{h_0}\left(1+|s|^{r_2-1}\right) \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \in \mathbb{R},$$

with $c_{f_0}, c_{h_0} > 0$ and $1 < r_1 < p^*$, $1 < r_2 < p_*$, where p^*, p_* denote the critical exponents of p given by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases} \quad \text{and} \quad p_* = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases}.$$

Setting $F_0(x,s) = \int_0^s f_0(x,t)dt$, $H_0(x,s) = \int_0^s h_0(x,t)dt$ we define the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ through

$$\varphi_0(u) = \int_\Omega G(\nabla u)dx - \int_\Omega F_0(x,u)dx - \int_{\partial\Omega} H_0(x,u)d\sigma.$$

The following result concerning local minimizers is originally due to Brezis-Nirenberg [2] and was extended by García Azorero-Peral Alonso-Manfredi [9], Motreanu-Papageorgiou [20], Winkert [24], and Khan-Motreanu [12].

Proposition 2.6 *Let the assumptions in H(a) be satisfied. If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, i.e., there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \textit{for all} \quad h \in C^1(\overline{\Omega}) \textit{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0,$$

then u_0 is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, i.e., there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \textit{for all } h \in W^{1,p}(\Omega) \textit{ with } \|h\|_{1,p} \leq \rho_1.$$

Proof The theorem follows directly from the abstract result obtained by Khan-Motreanu [12]. Indeed, let $X = C^1(\overline{\Omega})$, $Y = W^{1,p}(\Omega)$, and let

$$J(u) = \int_\Omega G(\nabla u)dx \quad \text{and} \quad E(u) = \int_\Omega F_0(x,u)dx + \int_{\partial\Omega} H_0(x,u)d\sigma.$$

Setting

$$\Phi(u) = \left(\|u\|_{r_1,\Omega}^{r_1} + \|u\|_{r_2,\partial\Omega}^{r_2}\right)^{\max(r_1,r_2)},$$

it is straightforward to verify that the assumptions in [12, Theorem 2.1] are satisfied. This completes the proof. □

Now, let $A : W^{1,p}(\Omega) \to \left(W^{1,p}(\Omega)\right)^*$ be the nonlinear map defined by

$$\langle A(u), v\rangle = \int_\Omega (a(\nabla u), \nabla v)_{\mathbb{R}^N} dx \quad \text{for all } u, v \in W^{1,p}(\Omega). \tag{2.2}$$

The next proposition gives the main properties of A (see, for example, Gasiński-Papageorgiou [10, p. 562]).

Proposition 2.7 *Let hypotheses H(a) be satisfied. Then* $A : W^{1,p}(\Omega) \to \left(W^{1,p}(\Omega)\right)^*$ *defined by* (2.2) *is continuous, monotone (hence maximal monotone) and of type* $(S)_+$, *i.e., if* $u_n \rightharpoonup u$ *in* $W^{1,p}(\Omega)$ *and* $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0$, *then* $u_n \to u$ *in* $W^{1,p}(\Omega)$.

Given $1 < r < \infty$, we denote by $\Delta_r : W^{1,r}(\Omega) \to \left(W^{1,r}(\Omega)\right)^*$ the r-Laplacian which is defined by

$$\langle \Delta_r u, v\rangle = \int_\Omega \|\nabla u\|_{\mathbb{R}^N}^{r-2} (\nabla u, \nabla v)_{\mathbb{R}^N}\, dx \quad \text{for all} \quad u, v \in W^{1,r}(\Omega).$$

If $r = 2$, then $\Delta_r = \Delta$ becomes the well-known Laplace operator and we have $\Delta \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$, where $\mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$ denotes the vector space of all bounded linear operators from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$.

A main role in our treatment plays the spectrum of the r-Laplacian with a Steklov boundary condition. To this end, we consider the following eigenvalue problem

$$\begin{aligned} -\Delta_r u &= -|u|^{r-2}u \qquad \text{in} \quad \Omega, \\ \|\nabla u\|_{\mathbb{R}^N}^{r-2} \frac{\partial u}{\partial n} &= \hat{\lambda}|u|^{r-2}u \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.3}$$

where $\partial u/\partial n$ is the outer normal derivative of u at $\partial\Omega$. A number $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of $\left(-\Delta_r, W^{1,r}(\Omega)\right)$ if problem (2.3) admits a nontrivial weak solution $\hat{u} \in W^{1,p}(\Omega)$ which is called an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. The set of eigenvalues is denoted by $\hat{\sigma}(r)$ which has a smallest element $\hat{\lambda}_1(r)$. The spectrum of (2.3) were intensively studied by Lê [13] and Martínez-Rossi [17] whereby the main facts read as follows:

- $\hat{\lambda}_1(r)$ is positive, isolated, and simple;
- $$\hat{\lambda}_1(r) = \inf_{u\in W^{1,p}(\Omega)} \left\{ \int_\Omega \|\nabla u\|_{\mathbb{R}^N}^r dx + \int_\Omega |u|^r dx : \|u\|_{r,\partial\Omega}^r = 1 \right\};$$
- $\hat{\sigma}(r)$ is closed.

We further point out that every eigenfunction corresponding to the first eigenvalue $\hat{\lambda}_1(r)$ does not change sign in $\overline{\Omega}$. In fact it turns out that every eigenfunction associated to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1(r)$ changes sign on $\partial\Omega$.

In what follows we denote by $\hat{u}_1(r)$ the normalized (i.e., $\|\hat{u}_1(r)\|_{r,\partial\Omega} = 1$) positive eigenfunction corresponding to $\hat{\lambda}_1(r)$. As shown in Lê [13], thanks to the nonlinear regularity theory and the nonlinear maximum principle, we can suppose that $\hat{u}_1(r) \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. Additionally, due to the fact that $\hat{\lambda}_1(r)$ is isolated, the second eigenvalue $\hat{\lambda}_2(r)$ is well-defined by

$$\hat{\lambda}_2(r) = \inf\left[\hat{\lambda} \in \hat{\sigma}(r) : \hat{\lambda} > \hat{\lambda}_1(r)\right].$$

Now, let $\partial B_1^{r,\partial\Omega} = \{u \in L^r(\partial\Omega) : \|u\|_{r,\partial\Omega} = 1\}$ and $S_r = W^{1,r}(\Omega) \cap \partial B_1^{r,\partial\Omega}$. Then, due to Martínez-Rossi [19], we have the following variational characterization of $\hat{\lambda}_2(r)$.

Proposition 2.8 *There holds*

$$\hat{\lambda}_2(r) = \inf_{\hat{\gamma}\in\hat{\Gamma}(r)} \max_{-1\le t\le 1} \left[\int_\Omega \|\nabla\hat{\gamma}(t)\|_{\mathbb{R}^N}^r dx + \int_\Omega |\hat{\gamma}(t)|^r dx \right],$$

where $\hat{\Gamma}(r) = \left\{\hat{\gamma} \in C([-1,1], S_r) : \hat{\gamma}(-1) = -\hat{u}_1(r), \hat{\gamma}(1) = \hat{u}_1(r)\right\}$.

Recall that if a functional satisfies the PS-condition (or the C-condition) and it is bounded below, then it is coercive (see Čaklović-Li-Willem [3] and Gasiński-Papageorgiou [11, p. 614]). The converse assertion is in general not true, but in our setting we can give a positive answer.

Indeed, let $\hat{f} : \Omega\times\mathbb{R} \to \mathbb{R}$, $\hat{h} : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ be Carathéodory functions satisfying

$$|\hat{f}(x,s)| \le c_{\hat{f}}\left(1 + |s|^{r_1-1}\right) \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R},$$
$$|\hat{h}(x,s)| \le c_{\hat{h}}\left(1 + |s|^{r_2-1}\right) \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \in \mathbb{R},$$

with $c_{\hat{f}}, c_{\hat{h}} > 0$, $1 < r_1 < p^*$, and $1 < r_2 < p_*$. We set $\hat{F}(x,s) = \int_0^s \hat{f}(x,t)dt$, $\hat{H}(x,s) = \int_0^s \hat{h}(x,t)dt$ and consider the C^1-functional $\hat{\varphi} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}(u) = \int_\Omega G(\nabla u)dx - \int_\Omega \hat{F}(x,u)dx - \int_{\partial\Omega} \hat{H}(x,u)d\sigma.$$

Proposition 2.9 *If $\hat{\varphi}$ is coercive, then it satisfies the PS-condition.*

Proof Let $(u_n)_{n\ge1} \subseteq W^{1,p}(\Omega)$ be a PS-sequence, that is

$$|\hat{\varphi}(u_n)| \le \hat{M} \quad \text{for some } \hat{M} > 0, \text{ for all } n \ge 1, \tag{2.4}$$
$$\left(\hat{\varphi}\right)'(u_n) \to 0 \quad \text{in } \left(W^{1,p}(\Omega)\right)^*. \tag{2.5}$$

Since $\hat{\varphi}$ is coercive and due to (2.4) we easily verify that $(u_n)_{n\ge1}$ is bounded in $W^{1,p}(\Omega)$. Because of that we may assume

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega) \quad u_n \to u \quad \text{in } L^p(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\partial\Omega). \tag{2.6}$$

The assertion in (2.5) implies that

$$\left| \int_\Omega (a(\nabla u_n), \nabla v)_{\mathbb{R}^N} dx - \int_\Omega \hat{f}(x,u_n)vdx - \int_{\partial\Omega} \hat{h}(x,u_n)vd\sigma \right| \le \varepsilon_n \|v\|_{1,p},$$

for all $v \in W^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$. Now, choosing $v = u_n - u$, passing to the limit as $n \to \infty$, and using the convergence properties in (2.6), we have

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = \lim_{n\to\infty} \int_\Omega (a(\nabla u_n), \nabla (u_n - u))_{\mathbb{R}^N}\, dx = 0,$$

which by the $(S)_+$-property of A (see Proposition 2.7) gives $u_n \to u$ in $W^{1,p}(\Omega)$ proving that $\hat{\varphi}$ satisfies the PS-condition. □

Finally, for $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. Recall that

$$u^\pm \in W^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

The Lebesgue measure on $\mathbb{R}^N$ is given by $|\cdot|_N$.

3 Three solutions depending on Steklov eigenvalues

We are now interested in the existence of weak solutions to Eq. (1.1) depending on Steklov eigenvalues of the q-Laplacian with $1 < q \le p < \infty$. In order to prove this we need some additional assumptions on the map $a : \mathbb{R}^N \to \mathbb{R}^N$.

H(a)$_1$: $a(\xi) = a_0\left(\|\xi\|_{\mathbb{R}^N}\right)\xi$ for all $\xi \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$, hypotheses H(a)$_1$(i)–(iii) are the same as the corresponding hypotheses H(a)(i)–(iii) and

(iv) if $G_0(t) = \int_0^t s a_0(s)ds$ for all $t > 0$, then $t \mapsto G_0\left(t^{\frac{1}{q}}\right)$ is convex in $(0, +\infty)$ and

$$\limsup_{t\to 0^+} \frac{G_0(t)}{t^q} < +\infty.$$

Remark 3.1 The examples presented in Sect. 2 still satisfy hypotheses H$(a)_1$. Note that by hypothesis H(a)$_1$(iv) we find $c_7 > 0$ such that

$$G(\xi) \le c_7\left(\|\xi\|^q_{\mathbb{R}^N} + \|\xi\|^p_{\mathbb{R}^N}\right) \quad \text{for all} \quad \xi \in \mathbb{R}^N. \tag{3.1}$$

The hypotheses on the Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ and the number χ read as follows.

(H1) f is bounded on bounded sets;

(H2) $\lim_{s\to\pm\infty} \dfrac{f(x,s)}{|s|^{p-2}s} = +\infty$ uniformly for a.a. $x \in \Omega$;

(H3) $\lim_{s\to 0} \dfrac{f(x,s)}{|s|^{p-2}s} = 0$ uniformly for a.a. $x \in \Omega$;

(H4) h is bounded on bounded sets;

(H5) $\lim_{s\to\pm\infty} \dfrac{h(x,s)}{|s|^{p-2}s} = +\infty$ uniformly for a.a. $x \in \partial\Omega$;

(H6) $\lim_{s\to 0} \dfrac{h(x,s)}{|s|^{q-2}s} = 0$ uniformly for a.a. $x \in \partial\Omega$;

(H7) h satisfies the condition

$$|h(x_1,s_1) - h(x_2,s_2)| \le L\Big[|x_1 - x_2|^{\alpha} + |s_1 - s_2|^{\alpha}\Big],$$

for all pairs (x_1,s_1), (x_2,s_2) in $\partial\Omega \times [-K,K]$, where K is a positive constant and $\alpha \in (0,1]$;

(H8) χ is a real fixed number such that

$$0 < \chi \begin{cases} < +\infty & \text{if } q < p \\ \le 2pc_7 & \text{if } q = p \end{cases},$$

where c_7 is the positive constant given in Remark 3.1.

Remark 3.2 Note that hypothesis (H7) is needed for the usage of the C^1-regularity results of Lieberman [15]. It is obvious that $s \mapsto \lambda|s|^{q-2}s$ fulfills condition (H7) for $\lambda > 0$ and $1 < q \le p$. We also point out that no growth condition is imposed on f, h and thanks to (H3), (H6) we easily verify that $f(x,0) = h(x,0) = 0$ for a.a. $x \in \Omega$, resp., $x \in \partial\Omega$. Hence, $u = 0$ is a solution of problem (1.1).

A function $u \in W^{1,p}(\Omega)$ is said to be a (weak) solution of (1.1) if it satisfies the equation

$$\int_{\Omega} (a(\nabla u), \nabla\varphi)_{\mathbb{R}^N}\, dx = \int_{\Omega} \Big(-\chi|u|^{q-2}u - f(x,u)\Big)\varphi dx + \int_{\partial\Omega} \Big(\lambda|u|^{q-2}u - h(x,u)\Big)\varphi d\sigma$$

for all test functions $\varphi \in W^{1,p}(\Omega)$ while $d\sigma$ denotes the usual $(N-1)$-surface measure.

The conditions in (H2), (H5) imply the existence of constants M_1, $M_2 = M_2(\lambda) > 1$ such that

$$\begin{aligned} f(x,s)s &\ge |s|^p \quad \text{for a.a. } x \in \Omega \text{ and all } |s| \ge M_1, \\ h(x,s)s &\ge \lambda|s|^p \quad \text{for a.a. } x \in \Omega \text{ and all } |s| \ge M_2. \end{aligned} \tag{3.2}$$

Let $M_3 = \max(M_1, M_2)$. Taking $\overline{u} \equiv \varsigma \in [M_3, +\infty)$ and applying (3.2), $q \le p$, and $M_3 > 1$, we conclude

$$0 \ge -f(x,\overline{u}) \quad \text{a.e. in } \Omega \quad \text{and} \quad 0 \ge \lambda\overline{u}^{q-1} - h(x,\overline{u}) \quad \text{a.e. in } \partial\Omega. \tag{3.3}$$

In the same way, choosing $\underline{v} \equiv -\varsigma$, we obtain

$$0 \le -f(x,\underline{v}) \quad \text{a.e. in } \Omega \quad \text{and} \quad 0 \le \lambda|\underline{v}|^{q-2}\underline{v} - h(x,\underline{v}) \quad \text{a.e. in } \partial\Omega.$$

Now, we introduce the truncation functions

$$\begin{aligned} b^+(x,s) &= \begin{cases} 0 & \text{if } s < 0 \\ -f(x,s) & \text{if } 0 \le s \le \overline{u}\,, \\ -f(x,\overline{u}) & \text{if } \overline{u} < s \end{cases} \\ b^+_\lambda(x,s) &= \begin{cases} 0 & \text{if } s < 0 \\ \lambda s^{q-1} - h(x,s) & \text{if } 0 \le s \le \overline{u}\,, \\ \lambda \overline{u}^{q-1} - h(x,\overline{u}) & \text{if } \overline{u} < s \end{cases} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} b^-(x,s) &= \begin{cases} -f(x,\underline{v}) & \text{if } s < \underline{v} \\ -f(x,s) & \text{if } \underline{v} \le s \le 0\,, \\ 0 & \text{if } 0 < s \end{cases} \\ b^-_\lambda(x,s) &= \begin{cases} \lambda |\underline{v}|^{q-2}\underline{v} - h(x,\underline{v}) & \text{if } s < \underline{v} \\ \lambda |s|^{q-2}s - h(x,s) & \text{if } \underline{v} \le s \le 0\,, \\ 0 & \text{if } 0 < s \end{cases} \end{aligned}$$

which are well-known to be Carathéodory functions. Setting $B^\pm(x,s) = \int_0^s b^\pm(x,t)dt$, $B^\pm_\lambda(x,s) = \int_0^s b^\pm_\lambda(x,t)dt$, we consider the C^1-functionals $\varphi^\pm_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi^\pm_\lambda(u) = \int_\Omega G(\nabla u)dx + \frac{\chi}{p}\int_\Omega |u|^p dx - \int_\Omega B^\pm(x,u)dx - \int_{\partial\Omega} B^\pm_\lambda(x,u)d\sigma.$$

Furthermore, we write $F(x,s) = \int_0^s f(x,t)dt$ and $H(x,s) = \int_0^s h(x,t)dt$. Recall that $\hat{u}_1(q) \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ denotes the normalized eigenfunction (i.e. $\|\hat{u}_1(q)\|_{q,\partial\Omega} = 1$) corresponding to the first eigenvalue $\hat{\lambda}_1(q)$ of the Steklov eigenvalue problem of the q-Laplacian given in (2.3).

We start with the existence of constant sign solutions to problem (1.1) provided $\lambda > 0$ is sufficiently large.

Proposition 3.3 *Let the assumptions in H(a)$_1$ and (H1)–(H8) be satisfied and suppose that*

$$\lambda > \begin{cases} q c_7 \hat{\lambda}_1(q) & \text{if } q < p \\ 2p c_7 \hat{\lambda}_1(p) & \text{if } q = p \end{cases} \tag{3.5}$$

with the positive constant c_7 given in Remark 3.1. Then problem (1.1) has at least two nontrivial constant sign solutions

$$u_0 \in [0,\overline{u}] \cap \operatorname{int}\left(C^1(\overline{\Omega})_+\right) \quad \text{and} \quad v_0 \in [\underline{v},0] \cap \left(-\operatorname{int}\left(C^1(\overline{\Omega})_+\right)\right).$$

Proof Let us begin the proof with the existence of the positive solution. By means of the truncation in (3.4) standard arguments ensure that φ_λ^+ is coercive and sequentially weakly lower semicontinuous. Therefore, the Weierstrass theorem yields the existence of $u_0 \in W^{1,p}(\Omega)$ such that

$$\varphi_\lambda^+(u_0) = \inf\left[\varphi_\lambda^+(u) : u \in W^{1,p}(\Omega)\right] = m_\lambda^+. \tag{3.6}$$

Given $\varepsilon_1, \varepsilon_2 > 0$, from Hypotheses (H3), (H6) we find $\delta_1 = \delta_1(\varepsilon_1), \delta_2 = \delta_2(\varepsilon_2) \in (0, \overline{u})$ such that

$$\begin{aligned} F(x,s) &\le \frac{\varepsilon_1}{p}|s|^p \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \le \delta_1, \\ H(x,s) &\le \frac{\varepsilon_2}{q}|s|^q \quad \text{for a.a. } x \in \partial\Omega \text{ and for all } |s| \le \delta_2. \end{aligned} \tag{3.7}$$

Let $\delta := \min(\delta_1, \delta_2)$. Since $\hat{u}_1(q) \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$, we may choose $t \in (0,1)$ small enough such that $t\hat{u}_1(q)(x) \in [0, \delta]$ for all $x \in \overline{\Omega}$. Because of (3.1), (3.4), (3.7) along with $\|\hat{u}_1(q)\|_{q,\partial\Omega} = 1$ and $\delta < \overline{u}$ it follows

$$\begin{aligned} \varphi_\lambda^+(t\hat{u}_1(q)) &= \int_\Omega G(\nabla(t\hat{u}_1(q)))dx + \frac{\chi}{p}\int_\Omega |t\hat{u}_1(q)|^p dx - \int_\Omega B^+(x, t\hat{u}_1(q))dx \\ &\quad - \int_{\partial\Omega} B_\lambda^+(x, t\hat{u}_1(q))d\sigma \\ &= \int_\Omega G(\nabla(t\hat{u}_1(q)))dx + \frac{t^p\chi}{p}\|\hat{u}_1(q)\|_{p,\Omega}^p + \int_\Omega F\left(x, t\hat{u}_1(q)\right)dx \\ &\quad - \frac{\lambda t^q}{q} + \int_{\partial\Omega} H\left(x, t\hat{u}_1(q)\right)d\sigma \\ &\le c_7\left(t^q\|\nabla(\hat{u}_1(q))\|_{q,\Omega}^q + t^p\|\nabla(\hat{u}_1(q))\|_{p,\Omega}^p\right) + \frac{t^p\chi}{p}\|\hat{u}_1(q)\|_{p,\Omega}^p \\ &\quad + \frac{\varepsilon_1 t^p}{p}\|\hat{u}_1(q)\|_{p,\Omega}^p - \frac{\lambda t^q}{q} + \frac{\varepsilon_2 t^q}{q} \\ &\le c_7\left(-t^q\|\hat{u}_1(q)\|_{q,\Omega}^q + t^q\hat{\lambda}_1(q) + t^p\|\nabla(\hat{u}_1(q))\|_{p,\Omega}^p\right) \\ &\quad + \frac{t^p\chi}{p}\|\hat{u}_1(q)\|_{p,\Omega}^p + \frac{\varepsilon_1 t^p}{p}\|\hat{u}_1(q)\|_{p,\Omega}^p - \frac{\lambda t^q}{q} + \frac{\varepsilon_2 t^q}{q} \\ &= t^q\left(\frac{-c_7 q}{q}\right)\|\hat{u}_1(q)\|_{q,\Omega}^q + t^q\left(\frac{qc_7\hat{\lambda}_1(q) - \lambda + \varepsilon_2}{q}\right) \\ &\quad + t^p\left(c_7\|\nabla(\hat{u}_1(q))\|_{p,\Omega}^p + \frac{\chi + \varepsilon_1}{p}\|\hat{u}_1(q)\|_{p,\Omega}^p\right). \end{aligned} \tag{3.8}$$

If $q < p$ we may choose $\varepsilon_1, \varepsilon_2$ such that

$$0 < \varepsilon_1 < \infty \quad \text{and} \quad 0 < \varepsilon_2 < \lambda - qc_7\hat{\lambda}_1(q)$$

(see (3.5)), then (3.8) becomes

$$\varphi_\lambda^+(t\hat{u}_1(q)) \leq -t^q M_4 + t^p M_5 \tag{3.9}$$

with some $M_4, M_5 > 0$. Since $q < p$, (3.9) implies

$$\varphi_\lambda^+(t\hat{u}_1(q)) < 0 \quad \text{for all sufficiently small } t > 0. \tag{3.10}$$

If $q = p$, (3.8) reduces to

$$\begin{aligned} \varphi_\lambda^+(t\hat{u}_1(p)) \leq t^p \left(\frac{-2pc_7 + \chi + \varepsilon_1}{p}\right) \|\hat{u}_1(p)\|_{p,\Omega}^p \\ + t^p \left(\frac{2pc_7\hat{\lambda}_1(p) - \lambda + \varepsilon_2}{p}\right). \end{aligned} \tag{3.11}$$

If $2pc_7 > \chi$ we may choose

$$0 < \varepsilon_1 < 2pc_7 - \chi \quad \text{and} \quad 0 < \varepsilon_2 < \lambda - 2pc_7\hat{\lambda}_1(p)$$

(see (H8) and (3.5)) to obtain again (3.10). Finally, if $2pc_7 = \chi$, (3.11) becomes

$$\varphi_\lambda^+(t\hat{u}_1(p)) \leq t^p \left(\frac{\varepsilon_1}{p}\|\hat{u}_1(p)\|_{p,\Omega}^p + \frac{2c_7 p\hat{\lambda}_1(p) - \lambda + \varepsilon_2}{p}\right).$$

Choosing $0 < \varepsilon_2 < \lambda - 2pc_7\hat{\lambda}_1(p)$ we find a constant $M_6 = M_6(\lambda) > 0$ such that

$$\varphi_\lambda^+(t\hat{u}_1(p)) \leq t^p \left(\frac{\varepsilon_1}{p}\|\hat{u}_1(p)\|_{p,\Omega}^p - M_6\right).$$

Taking $0 < \varepsilon_1 < \frac{M_6 p}{\|\hat{u}_1(p)\|_{p,\Omega}^p}$ provides inequality (3.10) again in this case. In summary the choices of $\varepsilon_1, \varepsilon_2$ above lead to (see also (3.6))

$$\varphi_\lambda^+(u_0) < 0 = \varphi_\lambda^+(0),$$

implying $u_0 \not\equiv 0$. Moreover, as u_0 is the global minimizer of φ_λ^+, there holds $\left(\varphi_\lambda^+\right)'(u_0) = 0$ meaning that

$$\begin{aligned} &\int_\Omega (a(\nabla u_0), \nabla\varphi)_{\mathbb{R}^N}\, dx + \chi \int_\Omega |u_0|^{p-2} u_0 \varphi dx \\ &\qquad = \int_\Omega b^+(x, u_0)\varphi dx + \int_{\partial\Omega} b_\lambda^+(x, u_0)\varphi d\sigma, \end{aligned} \tag{3.12}$$

for all $\varphi \in W^{1,p}(\Omega)$. We take $\varphi = -u_0^- \in W^{1,p}(\Omega)$ as test function in (3.12) and by virtue of Lemma 2.4(iii) in combination with the definition of the truncations (see (3.4)), we obtain

$$\min\left(\frac{c_2}{p-1}, \chi\right)\left(\|\nabla u_0^-\|_{p,\Omega}^p + \|u_0^-\|_{p,\Omega}^p\right) \le 0,$$

which means that $u_0 \ge 0$. Choosing $\varphi = (u_0 - \overline{u})^+ \in W^{1,p}(\Omega)$ in (3.12) and making use of (3.3) as well as (3.4), it follows

$$\begin{aligned} &\int_\Omega \left(a(\nabla u_0), \nabla(u_0 - \overline{u})^+\right)_{\mathbb{R}^N} dx + \chi \int_\Omega u_0^{p-1} (u_0 - \overline{u})^+ dx \\ &= \int_\Omega b^+(x, u_0)(u_0 - \overline{u})^+ dx + \int_{\partial\Omega} b_\lambda^+(x, u_0)(u_0 - \overline{u})^+ d\sigma \\ &= \int_\Omega (-f(x, \overline{u}))\,(u_0 - \overline{u})^+ dx + \int_{\partial\Omega} \left(\lambda \overline{u}^{q-1} - h(x, \overline{u})\right)(u_0 - \overline{u})^+ d\sigma \\ &\le 0. \end{aligned}$$

This gives, due to Lemma 2.4(iii),

$$\begin{aligned} 0 &\ge \int_{\{u_0 > \overline{u}\}} (a(\nabla u_0), \nabla u_0)_{\mathbb{R}^N}\, dx + \chi \int_{\{u_0 > \overline{u}\}} u_0^{p-1}\,(u_0 - \overline{u})\, dx \\ &\ge \frac{c_2}{p-1} \int_{\{u_0 > \overline{u}\}} \|\nabla u_0\|_{\mathbb{R}^N}^p\, dx + \int_{\{u_0 > \overline{u}\}} (u_0 - \overline{u})^p\, dx \\ &\ge \int_\Omega \left((u_0 - \overline{u})^+\right)^p dx \\ &\ge 0. \end{aligned}$$

Hence $|\{u_0 > \overline{u}\}|_N = 0$, that is $u_0 \leq \overline{u}$. We conclude that $u_0 \in [0, \overline{u}]$ with $u_0 \not\equiv 0$. Then, by means of the truncations in (3.4), Eq. (3.12) becomes

$$\int_\Omega (a(\nabla u_0), \nabla\varphi)_{\mathbb{R}^N}\, dx + \chi \int_\Omega u_0^{p-1}\varphi dx = \int_\Omega (-f(x,u_0))\,\varphi dx + \int_{\partial\Omega} \left(\lambda u_0^{q-1} - h(x,u_0)\right)\varphi d\sigma$$

which means that $u_0 \in W^{1,p}(\Omega)$ solves the problem

$$\begin{aligned} -\operatorname{div} a(\nabla u_0) &= -\chi u_0^{p-1} - f(x,u_0) && \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n_a} &= \lambda u_0^{q-1} - h(x,u_0) && \text{on} \quad \partial\Omega. \end{aligned} \tag{3.13}$$

Since $u_0 \in [0, \overline{u}]$ we have $u_0 \in L^\infty(\Omega)$ (see also Winkert-Zacher [26, Corollary 1.2]) and from the regularity results of Lieberman [15] it follows that $u_0 \in C^1(\overline{\Omega})\backslash\{0\}$. Taking into account (H1), (H3) we find a constant $M_7 > 0$ such that

$$f(x,s) \leq M_7 s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \leq s \leq \overline{u}. \tag{3.14}$$

Combining (3.13) and (3.14) yields

$$\operatorname{div} a(\nabla u_0) \leq (\chi + M_7)\, u_0^{p-1} \quad \text{a.e. in} \quad \Omega.$$

Now, we may apply the strong maximum principle (see Pucci-Serrin [21, Theorem 2.5.1]) to obtain $u_0(x) > 0$ for all $x \in \Omega$.

Let $x_0 \in \partial\Omega$ be such that $u_0(x_0) = 0$. Applying the boundary point lemma (see again Pucci-Serrin [21, Theorem 5.5.1]) gives

$$\frac{\partial u_0}{\partial n_a}(x_0) = a_0\left(\|\nabla u_0\|_{\mathbb{R}^N}\right)\frac{\partial u_0}{\partial n}(x_0) < 0, \tag{3.15}$$

where $(\partial u_0/\partial n)(x_0)$ stands for the outer normal derivative of u_0 at $x_0 \in \partial\Omega$. Since $h(x_0, u_0(x_0)) = h(x_0, 0) = 0$ we get a contradiction from (3.13) and (3.15). Hence, $u_0(x) > 0$ for all $x \in \overline{\Omega}$ and consequently $u_0 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. That finishes the first part of the theorem.

The second assertion can be shown similarly using φ_λ^- instead of φ_λ^+ to obtain the existence of a nontrivial negative solution $v_0 \in [\underline{v}, 0] \cap \left(-\operatorname{int}\left(C^1(\overline{\Omega})_+\right)\right)$. □

Now we are going to prove the existence of extremal constant sign solutions of (1.1) provided $\lambda > 0$ is large enough as before.

To this end, let $\mathcal{S}_+(\lambda)(\mathcal{S}_-(\lambda))$ be the set of all nontrivial positive (negative) solutions of problem (1.1). Thanks to the monotonicity of a (see Lemma 2.4(i)) one can show that $\mathcal{S}_+(\lambda)$ $(\mathcal{S}_-(\lambda))$ is downward (upward) directed, that means if $u_1, u_2 \in \mathcal{S}_+(\lambda)$

$(\mathcal{S}_-(\lambda))$, then there is an element $\hat{v} \in \mathcal{S}_+(\lambda)$ $(\mathcal{S}_-(\lambda))$ such that $\hat{v} \leq (\geq) u_1$, $\hat{v} \leq (\geq) u_2$. Therefore, without loss of generality, we can focus on the sets

$$\hat{\mathcal{S}}_+(\lambda) = \mathcal{S}_+(\lambda) \cap [0, \overline{u}], \qquad \hat{\mathcal{S}}_-(\lambda) = \mathcal{S}_-(\lambda) \cap [\underline{v}, 0].$$

As a consequence of Proposition 3.3 we know that both sets are nonempty, i.e., $\hat{S}_+(\lambda) \neq \emptyset$ and $\hat{S}_-(\lambda) \neq \emptyset$. We can further suppose, without loss of generality, that

$$\begin{aligned} |f(x,s)| &\leq M_8 \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R}, \\ |h(x,s)| &\leq M_9 \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \in \mathbb{R}, \end{aligned} \tag{3.16}$$

with positive constants M_8, M_9 which can be seen by truncation of $f(x,\cdot), h(x,\cdot)$ at $\underline{v}$ (from below) and $\overline{u}$ (from above) combined with (H1), (H4). Then, taking into account hypotheses (H3), (H6) along with (3.16) we find for given $\varepsilon_1, \varepsilon_2 > 0$ and $r_1 \in (p, p^*), r_2 \in (p, p_*)$ numbers $M_{10} = M_{10}(\varepsilon_1, r_1), M_{11} = M_{11}(\varepsilon_2, r_2) > 0$ such that

$$\begin{aligned} f(x,s)s &\leq \varepsilon_1 |s|^p + M_{10}|s|^{r_1} \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R}, \\ h(x,s)s &\leq \varepsilon_2 |s|^q + M_{11}|s|^{r_2} \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R}. \end{aligned} \tag{3.17}$$

In order to prove the existence of a smallest positive and a greatest negative solution to (1.1) we will consider an auxiliary problem. To this end, let $\lambda > 0, \varepsilon_1 > 0, \varepsilon_2 \in (0, \lambda)$ and consider the subsequent equation

$$\begin{aligned} -\operatorname{div}(a(\nabla u)) &= -(\chi + \varepsilon_1)|u|^{p-2}u - M_{10}|u|^{r_1-2}u \qquad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n_a} &= (\lambda - \varepsilon_2)|u|^{q-2}u - M_{11}|u|^{r_2-2}u \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.18}$$

We are going to prove the uniqueness of constant sign solutions of problem (3.18).

Proposition 3.4 *Let hypotheses H(a)$_1$ and (H8) be satisfied and suppose*

$$\lambda > \begin{cases} q c_7 \hat{\lambda}_1(q) & \text{if } q < p, \\ 2p c_7 \hat{\lambda}_1(p) & \text{if } q = p. \end{cases}$$

Then, problem (3.18) has a unique positive solution $u_* \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ and a unique negative solution $v_* \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$.

Proof Due to the oddness of (3.18) it suffices to prove the existence of a unique positive solution $u_* \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$, the existence of a unique negative solution follows directly by setting $v_* = -u_* \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$.

Let $\Psi_\lambda^+ : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\Psi_\lambda^+(u) = \int_\Omega G(\nabla u)dx + \frac{\chi}{p}\|u\|_{p,\Omega}^p + \frac{\varepsilon_1}{p}\|u^+\|_{p,\Omega}^p + \frac{M_{10}}{r_1}\|u^+\|_{r_1,\Omega}^{r_1} - \frac{\lambda-\varepsilon_2}{q}\|u^+\|_{q,\partial\Omega}^q + \frac{M_{11}}{r_2}\|u^+\|_{r_2,\partial\Omega}^{r_2}.$$

Since $q \le p < r_1 < p^*, q \le p < r_2 < p_*$ we note that Ψ_λ^+ is coercive and sequentially weakly lower semicontinuous. Hence, its global minimizer $u_* \in W^{1,p}(\Omega)$ exists and as in Proposition 3.3, the choice of $\lambda > 0$ yields

$$\Psi_\lambda^+(u_*) < 0 = \Psi_\lambda^+(0)$$

guaranteeing $u_* \neq 0$. Since u_* is a global minimizer of Ψ_λ^+ we have $\left(\psi_\lambda^+\right)'(u_*) = 0$, that is

$$\begin{aligned}
&\int_\Omega (a(\nabla u_*), \nabla\varphi)_{\mathbb{R}^N}\, dx + \chi\int_\Omega |u_*|^{p-2}u_*\varphi dx \\
&= -\varepsilon_1\int_\Omega \left(u_*^+\right)^{p-1}\varphi dx - M_{10}\int_\Omega \left(u_*^+\right)^{r_1-1}\varphi dx \\
&\quad + (\lambda-\varepsilon_2)\int_{\partial\Omega}\left(u_*^+\right)^{q-1}\varphi d\sigma - M_{11}\int_{\partial\Omega}\left(u_*^+\right)^{r_2-1}\varphi d\sigma,
\end{aligned} \tag{3.19}$$

for all $\varphi \in W^{1,p}(\Omega)$. Taking $\varphi = -u_*^- \in W^{1,p}(\Omega)$ and applying Lemma 2.4(iii), we get $u_* \ge 0$ (cf. the proof of Proposition 3.3). Then, (3.19) becomes

$$\begin{aligned}
&\int_\Omega (a(\nabla u_*), \nabla\varphi)_{\mathbb{R}^N}\, dx + \chi\int_\Omega u_*^{p-1}\varphi dx \\
&= -\varepsilon_1\int_\Omega u_*^{p-1}\varphi dx - M_{10}\int_\Omega u_*^{r_1-1}\varphi dx - (\lambda-\varepsilon_2)\int_{\partial\Omega} u_*^{q-1}\varphi d\sigma - M_{11}\int_{\partial\Omega} u_*^{r_2-1}\varphi d\sigma,
\end{aligned}$$

meaning that u_* is a nontrivial positive solution of (3.18). Moreover, the nonlinear regularity theory (see Winkert-Zacher [26] and Lieberman [15]) combined with the nonlinear maximum principle (see Pucci-Serrin [21]) yields $u_* \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ (similar to the proof of Proposition 3.3).

We are done with the proof provided u_* is shown to be the unique positive solution of (3.18). Let $\Upsilon_+ : L^1(\Omega) \to \mathbb{R} \cup \{\infty\}$ be the integral functional defined by

$$\Upsilon_+(u) = \begin{cases} \displaystyle\int_\Omega G\left(\nabla u^{\frac{1}{q}}\right)dx + \frac{M_{11}}{r_2}\int_{\partial\Omega}|u|^{\frac{r_2}{q}}d\sigma & \text{if } u \ge 0, u^{\frac{1}{q}} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let u_1, u_2 be in the domain of Υ_+, i.e. $u_1, u_2 \in \mathrm{dom}(\Upsilon_+) = \left\{u \in L^1(\Omega) : \Upsilon_+(u) < +\infty\right\}$ and let further $u = (tu_1 + (1-t)u_2)^{\frac{1}{q}}$ with $t \in [0, 1]$. Applying Lemma 1 in Díaz-Saá [4] there holds

$$\|\nabla u(x)\|_{\mathbb{R}^N} \leq \left(t \left\|\nabla u_1(x)^{\frac{1}{q}}\right\|_{\mathbb{R}^N}^q + (1-t)\left\|\nabla u_2(x)^{\frac{1}{q}}\right\|_{\mathbb{R}^N}^q\right)^{\frac{1}{q}} \quad \text{a.e. in } \Omega.$$

Since G_0 is increasing and thanks to condition H$(a)_1$(iv) it follows

$$\begin{aligned}
&G_0\left(\|\nabla u(x)\|_{\mathbb{R}^N}\right)\\
&\leq G_0\left(\left(t\left\|\nabla u_1(x)^{\frac{1}{q}}\right\|_{\mathbb{R}^N}^q + (1-t)\left\|\nabla u_2(x)^{\frac{1}{q}}\right\|_{\mathbb{R}^N}^q\right)^{\frac{1}{q}}\right)\\
&\leq tG_0\left(\left\|\nabla u_1(x)^{\frac{1}{q}}\right\|_{\mathbb{R}^N}\right) + (1-t)G_0\left(\left\|\nabla u_2(x)^{\frac{1}{q}}\right\|_{\mathbb{R}^N}\right) \quad \text{a.e. in } \Omega.
\end{aligned}$$

By definition $G(\xi) = G_0(\|\xi\|)$ for all $\xi \in \mathbb{R}^N$, hence

$$G(\nabla u(x)) \leq tG\left(\nabla u_1(x)^{\frac{1}{q}}\right) + (1-t)G\left(\nabla u_2(x)^{\frac{1}{q}}\right) \quad \text{a.e. in } \Omega.$$

Therefore, Υ_+ is convex and due to Fatou's lemma it is also lower semicontinuous.

Now, taking two positive solutions $u, v \in W^{1,p}(\Omega)$ of (3.18) and recalling that $u, v \in \mathrm{int}\left(C^1(\overline{\Omega})_+\right)$ (see the first part of the proof) we have $u, v \in \mathrm{dom}(\Upsilon_+)$. For $h \in C^1(\overline{\Omega})$ and $t \in (0, 1)$ sufficiently small we see that $u^q + th$, $v^q + th \in \mathrm{int}\left(C^1(\overline{\Omega})_+\right)$. Thus, Υ_+ is Gateaux differentiable at u^q and v^q in the direction h. Applying the chain rule and the nonlinear Green's identity (see, for example, Gasiński-Papageorgiou [11, p. 210]) yields

$$\Upsilon_+'\left(u^q\right)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(\nabla u)}{u^{q-1}}h dx + \frac{\lambda - \varepsilon_2}{q}\int_{\partial\Omega} h d\sigma, \tag{3.20}$$

$$\Upsilon_+'\left(v^q\right)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(\nabla v)}{v^{q-1}}h dx + \frac{\lambda - \varepsilon_2}{q}\int_{\partial\Omega} h d\sigma. \tag{3.21}$$

Since Υ_+' is monotone (follows from the convexity of Υ_+) and thanks to (3.20) as well as (3.21), we obtain

$$\begin{aligned}
0 &\leq \left\langle \Upsilon_+'\left(u^q\right) - \Upsilon_+'\left(v^q\right), u^q - v^q\right\rangle_{L^1(\Omega)}\\
&= \frac{1}{q}\int_\Omega \left(\frac{-\operatorname{div} a(\nabla u)}{u^{q-1}} - \frac{-\operatorname{div} a(\nabla v)}{v^{q-1}}\right)\left(u^q - v^q\right) dx
\end{aligned}$$

$$= \frac{1}{q} \int_\Omega \left(\frac{-(\chi+\varepsilon_1)u^{p-1} - M_{10}u^{r_1-1}}{u^{q-1}} - \frac{-(\chi+\varepsilon_1)v^{p-1} - M_{10}v^{r_1-1}}{v^{q-1}} \right) \left(u^q - v^q\right) dx$$

$$= \frac{M_{10}}{q} \int_\Omega \left(v^{r_1-q} - u^{r_1-q}\right)\left(u^q - v^q\right) dx + \frac{\chi+\varepsilon_1}{q} \int_\Omega \left(v^{p-q} - u^{p-q}\right)\left(u^q - v^q\right) dx.$$

Since $s \mapsto s^{r_1-q}$ and $s \mapsto s^{p-q}$ are strictly increasing in $(0,\infty)$ we obtain that $u = v$ and therefore, $u_* \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ is the unique positive solution of (3.18). □

With the aid of these solutions obtained in the last proposition we are now in the position to prove the existence of extremal constant sign solutions of our original problem (1.1) provided $\lambda > 0$ is sufficiently large.

Proposition 3.5 *Let the assumptions in H(a)$_1$ and (H1)–(H8) be satisfied and assume*

$$\lambda > \begin{cases} qc_7\hat{\lambda}_1(q) & \text{if } q < p, \\ 2pc_7\hat{\lambda}_1(p) & \text{if } q = p. \end{cases}$$

Then problem (1.1) has a smallest positive solution $u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ and a greatest negative solution $v_- \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$.

Proof As mentioned before it is enough to prove the existence of these extremal solutions in the sets $\hat{\mathcal{S}}_+(\lambda) = \mathcal{S}_+(\lambda) \cap [0,\overline{u}] \subseteq \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ and $\hat{\mathcal{S}}_-(\lambda) = \mathcal{S}_-(\lambda) \cap [\underline{v},0] \subseteq -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$.

First, we are going to prove that $u_* \leq u$ for all $u \in \hat{S}_+(\lambda)$. For this purpose, let $\hat{v} \in \hat{S}_+(\lambda)$ and define the Carathéodory functions $\zeta^+ : \Omega \times \mathbb{R} \to \mathbb{R}$, $\zeta_\lambda^+ : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ through

$$\zeta^+(x,s) = \begin{cases} 0 & \text{if } s < 0 \\ -\varepsilon_1 s^{p-1} - M_{10}s^{r_1-1} & \text{if } 0 \leq s \leq \hat{v}(x), \\ -\varepsilon_1 \hat{v}(x)^{p-1} - M_{10}\hat{v}(x)^{r_1-1} & \text{if } \hat{v}(x) < s \end{cases} \tag{3.22}$$

and

$$\zeta_\lambda^+(x,s) = \begin{cases} 0 & \text{if } s < 0 \\ (\lambda-\varepsilon_2)s^{q-1} - M_{11}s^{r_2-1} & \text{if } 0 \leq s \leq \hat{v}(x). \\ (\lambda-\varepsilon_2)\hat{v}(x)^{q-1} - M_{11}\hat{v}(x)^{r_2-1} & \text{if } \hat{v}(x) < s \end{cases} \tag{3.23}$$

Moreover, we define the C^1-functional $\Xi_\lambda^+ : W^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\Xi_\lambda^+(u) = \int_\Omega G(\nabla u)dx + \frac{\chi}{p}\int_\Omega |u|^p dx - \int_\Omega Z^+(x,u)dx - \int_{\partial\Omega} Z_\lambda^+(x,u)d\sigma$$

with $Z^+(x,s)=\int_0^s \zeta^+(x,t)dt$ and $Z_\lambda^+(x,s)=\int_0^s \zeta_\lambda^+(x,t)dt$. Thanks to the truncations we easily verify that Ξ_λ^+ is coercive and sequentially weakly lower semicontinuous. Hence, there exists the global minimizer of Ξ_λ^+ on $W^{1,p}(\Omega)$, i.e.

$$\Xi_\lambda^+(\hat{u}_*)=\inf\left[\Xi_\lambda^+(u):u\in W^{1,p}(\Omega)\right]. \tag{3.24}$$

As in the proof of Proposition 3.3 we can show that

$$\Xi_\lambda^+\left(\hat{u}_*\right)<0=\Xi_\lambda^+(0),$$

meaning $\hat{u}_*\neq 0$. Moreover, due to (3.24), there holds

$$\begin{aligned}&\int_\Omega \left(a(\nabla\hat{u}_*),\nabla\varphi\right)_{\mathbb{R}^N}dx+\chi\int_\Omega |\hat{u}_*|^{p-2}\hat{u}_*\varphi dx\\&=\int_\Omega \zeta^+\left(x,\hat{u}_*\right)\varphi dx+\int_{\partial\Omega}\zeta_\lambda^+\left(x,\hat{u}_*\right)\varphi d\sigma,\end{aligned} \tag{3.25}$$

for all $\varphi\in W^{1,p}(\Omega)$. Taking $\varphi=-\hat{u}_*^-\in W^{1,p}(\Omega)$ in (3.25) and applying Lemma 2.4(iii) we derive

$$\min\left(\frac{c_2}{p-1},\chi\right)\left(\|\nabla\hat{u}_*^-\|_{p,\Omega}^p+\|\hat{u}_*^-\|_{p,\Omega}^p\right)\leq 0,$$

which implies $\hat{u}_*\geq 0$. Since $\hat{v}$ is a positive solution of (1.1) it satisfies

$$\begin{aligned}&\int_\Omega \left(a\left(\nabla\hat{v}\right),\nabla\varphi\right)_{\mathbb{R}^N}dx+\chi\int_\Omega \hat{v}^{p-1}\varphi dx\\&=\int_\Omega\left(-f\left(x,\hat{v}\right)\right)\varphi dx+\int_{\partial\Omega}\left(\lambda\hat{v}^{q-1}-h\left(x,\hat{v}\right)\right)\varphi d\sigma,\end{aligned} \tag{3.26}$$

for all $\varphi\in W^{1,p}(\Omega)$. Choosing $\left(\hat{u}_*-\hat{v}\right)^+\in W^{1,p}(\Omega)$ in (3.25) and (3.26), subtracting (3.26) from (3.25), and making use of (3.17), (3.22) as well as (3.23) we derive

$$\begin{aligned}&\int_\Omega\left(a\left(\nabla\hat{u}_*\right)-a\left(\nabla\hat{v}\right),\nabla\left(\hat{u}_*-\hat{v}\right)^+\right)_{\mathbb{R}^N}dx+\chi\int_\Omega\left((\hat{u}_*)^{p-1}-\hat{v}^{p-1}\right)\left(\hat{u}_*-\hat{v}\right)^+dx\\&=\int_\Omega\left(\zeta^+(x,\hat{u}_*)+f\left(x,\hat{v}\right)\right)\left(\hat{u}_*-\hat{v}\right)^+dx\\&\quad+\int_{\partial\Omega}\left(\zeta_\lambda^+(x,\hat{u}_*)-\lambda\hat{v}^{q-1}+h\left(x,\hat{v}\right)\right)\left(\hat{u}_*-\hat{v}\right)^+d\sigma\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \left(-\varepsilon_1 \hat{v}^{p-1} - M_{10}\hat{v}^{r_1-1} + f\left(x,\hat{v}\right)\right)\left(\hat{u}_* - \hat{v}\right)^+ dx \\
&\quad + \int_{\partial\Omega} \left((\lambda-\varepsilon_2)\,\hat{v}^{q-1} - M_{11}\hat{v}^{r_2-1} - \lambda\hat{v}^{q-1} + h\left(x,\hat{v}\right)\right)\left(\hat{u}_* - \hat{v}\right)^+ d\sigma \\
&\leq 0.
\end{aligned}
$$

This implies for $\hat{u}_* > \hat{v}$, due to Lemma 2.4(i),

$$
\begin{aligned}
0 &\geq \int_\Omega \left(a\left(\nabla\hat{u}_*\right) - a\left(\nabla\hat{v}\right), \nabla\left(\hat{u}_* - \hat{v}\right)^+\right)_{\mathbb{R}^N} dx \\
&\quad + \chi \int_\Omega \left(\left(\hat{u}_*\right)^{p-1} - \hat{v}^{p-1}\right)\left(\hat{u}_* - \hat{v}\right)^+ dx \\
&> 0,
\end{aligned}
$$

which is a contradiction. Therefore $\hat{u}_* \leq \hat{v}$. To sum up, we have shown that $\hat{u}_* \in \left[0, \hat{v}\right]$ and $\hat{u}_* \not\equiv 0$. Then, by definition of the truncations, we obtain that $\hat{u}_*$ is a positive solution of (3.18) which by Proposition 3.4 implies that $\hat{u}_* = u_* \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. Hence

$$
u_* \leq u \qquad \text{for all} \quad u \in \hat{\mathcal{S}}_+(\lambda). \tag{3.27}
$$

Now let $\mathcal{C} \subseteq \hat{\mathcal{S}}_+(\lambda)$ be a chain, that means, a totally ordered subset of $\hat{\mathcal{S}}_+(\lambda)$. Then there exists a sequence $(u_n)_{n\geq 1} \subseteq \hat{\mathcal{S}}_+(\lambda)$ (see Dunford-Schwartz [5, p. 336]) such that $\inf \mathcal{C} = \inf_{n\geq 1} u_n$. As u_n is a positive solution of (1.1) we have

$$
\begin{aligned}
&\int_\Omega \left(a(\nabla u_n), \nabla\varphi\right)_{\mathbb{R}^N} dx \\
&\quad = \int_\Omega \left(-\chi u_n^{p-1} - f(x,u_n)\right)\varphi dx + \int_{\partial\Omega} \left(\lambda u_n^{q-1} - h(x,u_n)\right)\varphi d\sigma,
\end{aligned} \tag{3.28}
$$

for all $\varphi \in W^{1,p}(\Omega)$ with $u_* \leq u_n \leq \overline{u}$ for all $n \geq 1$ (see (3.27)). Since f and h are bounded on bounded sets we obtain the boundedness of u_n in $W^{1,p}(\Omega)$. Therefore, we may assume that

$$
u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega), \qquad u_n \to u \text{ in } L^p(\Omega), \qquad u_n \to u \text{ in } L^p(\partial\Omega). \tag{3.29}
$$

Taking $\varphi = u_n - u \in W^{1,p}(\Omega)$ in (3.28) and passing to the limit as $n \to \infty$, one gets, thanks to the convergence properties in (3.29),

$$
\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = \lim_{n\to\infty} \int_\Omega \left(a(\nabla u_n), \nabla(u_n - u)\right)_{\mathbb{R}^N} dx = 0. \tag{3.30}
$$

Since A satisfies the (S_+)-property (see Proposition 2.7), (3.29) and (3.30) imply $u_n \to u$ in $W^{1,p}(\Omega)$. Using this fact we can pass again to the limit in (3.28) which gives

$$\int_\Omega (a(\nabla u), \nabla \varphi)_{\mathbb{R}^N} \, dx = \int_\Omega (-\chi u^{p-1} - f(x,u))\varphi dx + \int_{\partial\Omega} (\lambda u^{q-1} - h(x,u))\varphi d\sigma$$

with $u_* \leq u \leq \overline{u}$. That means $u \in \hat{\mathcal{S}}_+(\lambda)$ and $u = \inf \mathcal{C}$. Then, the Kuratowski-Zorn Lemma implies that $\hat{\mathcal{S}}_+(\lambda)$ has a minimal element $u_+ \in \hat{\mathcal{S}}_+(\lambda)$ and since $\hat{\mathcal{S}}_+(\lambda)$ is downward directed, we infer that u_+ is the smallest positive solution of (1.1).

The existence of a greatest negative solution $v_- \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ of (1.1) can be shown in the same way, working with the set $\hat{\mathcal{S}}_-(\lambda)$ instead of $\hat{\mathcal{S}}_+(\lambda)$. The proof is complete. □

Now, we are going to prove the existence of a nontrivial solution y_0 of (1.1) which turns out to be a sign changing solution.

Proposition 3.6 *If hypotheses $H(a)_1$ and (H1)–(H8) hold and if*

$$\lambda > \begin{cases} q c_7 \hat{\lambda}_2(q) & \text{if } q < p \\ 2p c_7 \hat{\lambda}_2(p) & \text{if } q = p \end{cases}$$

is satisfied, then problem (1.1) has a nodal solution $y_0 \in C^1\left(\overline{\Omega}\right)$.

Proof Recall that $u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ and $v_- \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ are the two extremal constant sign solutions of (1.1) obtained by Proposition 3.5. Let $\vartheta : \Omega \times \mathbb{R} \to \mathbb{R}$, $\vartheta_\lambda : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, be truncation functions defined by

$$\vartheta(x,s) = \begin{cases} -f(x, v_-(x)) & \text{if } s < v_-(x) \\ -f(x,s) & \text{if } v_-(x) \leq s \leq u_+(x) \\ -f(x, u_+(x)) & \text{if } u_+(x) < s \end{cases} \tag{3.31}$$

and

$$\vartheta_\lambda(x,s) = \begin{cases} \lambda |v_-(x)|^{q-2} v_-(x) - h(x, v_-(x)) & \text{if } s < v_-(x) \\ \lambda |s|^{q-2} s - h(x,s) & \text{if } v_-(x) \leq s \leq u_+(x) \\ \lambda u_+(x)^{q-1} - h(x, u_+(x)) & \text{if } u_+(x) < s \end{cases}. \tag{3.32}$$

Furthermore, let $\vartheta^{\pm}(x,s)=\vartheta\left(x,\pm s^{\pm}\right)$, $\vartheta_{\lambda}^{\pm}(x,s)=\vartheta_{\lambda}\left(x,\pm s^{\pm}\right)$ and define

$$\Theta(x,s)=\int_0^s \vartheta(x,t)\,dt, \qquad \Theta_{\lambda}(x,s)=\int_0^s \vartheta_{\lambda}(x,t)\,dt,$$
$$\Theta^{\pm}(x,s)=\int_0^s \vartheta^{\pm}(x,t)\,dt, \qquad \Theta_{\lambda}^{\pm}(x,s)=\int_0^s \vartheta_{\lambda}^{\pm}(x,t)\,dt.$$

We consider the C^1-functionals $\Phi_{\lambda}, \Phi_{\lambda}^{\pm}: W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\Phi_{\lambda}(u)=\int_{\Omega} G(\nabla u)\,dx+\frac{\chi}{p}\int_{\Omega}|u|^p dx-\int_{\Omega}\Theta(x,u)dx-\int_{\partial\Omega}\Theta_{\lambda}(x,u)d\sigma,$$
$$\Phi_{\lambda}^{\pm}(u)=\int_{\Omega} G(\nabla u)dx+\frac{\chi}{p}\int_{\Omega}|u|^p dx-\int_{\Omega}\Theta^{\pm}(x,u)dx-\int_{\partial\Omega}\Theta_{\lambda}^{\pm}(x,u)d\sigma.$$

First, we will prove that

$$K_{\Phi_{\lambda}}\subseteq[v_-,u_+], \quad K_{\Phi_{\lambda}^{+}}=\{0,u_+\}, \quad K_{\Phi_{\lambda}^{-}}=\{v_-,0\}. \tag{3.33}$$

To this end, let $u\in K_{\Phi_{\lambda}}$, that is

$$\begin{aligned}&\int_{\Omega}(a(\nabla u),\nabla\varphi)_{\mathbb{R}^N}dx+\chi\int_{\Omega}|u|^{p-2}u\varphi dx\\&=\int_{\Omega}\vartheta(x,u)\varphi dx+\int_{\partial\Omega}\vartheta_{\lambda}(x,u)\varphi d\sigma \quad \text{for all} \quad \varphi\in W^{1,p}(\Omega).\end{aligned} \tag{3.34}$$

Since u_+ is a positive solution of (1.1), we have

$$\begin{aligned}&\int_{\Omega}(a(\nabla u_+),\nabla\varphi)_{\mathbb{R}^N}dx+\chi\int_{\Omega}u_+^{p-1}\varphi dx\\&=\int_{\Omega}(-f(x,u_+))\varphi dx+\int_{\partial\Omega}\left(\lambda u_+^{p-1}-h(x,u_+)\right)\varphi d\sigma,\end{aligned} \tag{3.35}$$

for all $\varphi \in W^{1,p}(\Omega)$. Choosing $\varphi = (u-u_+)^+ \in W^{1,p}(\Omega)$ in (3.34) and (3.35) and subtracting (3.35) from (3.34) results in

$$\begin{aligned}
&\int_\Omega \left(a(\nabla u)-a(\nabla u_+), \nabla (u-u_+)^+\right)_{\mathbb{R}^N} dx + \chi \int_\Omega \left(|u|^{p-2}u - u_+^{p-1}\right)(u-u_+)^+ dx \\
&= \int_\Omega (\vartheta(x,u) + f(x,u_+))(u-u_+)^+ dx \\
&\quad + \int_{\partial\Omega} \left(\vartheta_\lambda(x,u) - \lambda u_+^{p-1} + h(x,u_+)\right)(u-u_+)^+ d\sigma \\
&= 0,
\end{aligned}$$

due to the definition of the truncations in (3.31) and (3.32). Since $a : \mathbb{R}^N \to \mathbb{R}^N$ is strictly monotone (see Lemma 2.4(i)), we derive for $u > u_+$

$$\begin{aligned}
0 &= \int_\Omega \left(a(\nabla u)-a(\nabla u_+), \nabla (u-u_+)^+\right)_{\mathbb{R}^N} + \chi \int_\Omega \left(|u|^{p-2}u - u_+^{p-1}\right)(u-u_+)^+ dx \\
&> 0,
\end{aligned}$$

which is a contradiction. This gives $u \le u_+$.

Acting on (3.34) and (3.35) with $\varphi = (v_- - u)^+ \in W^{1,p}(\Omega)$ and subtracting again we obtain $v_- \le u$. Hence

$$K_{\Phi_\lambda} \subseteq [v_-, u_+].$$

Following the same ideas we can prove that

$$K_{\Phi_\lambda^+} \subseteq [0, u_+] \quad \text{and} \quad K_{\Phi_\lambda^-} \subseteq [v_-, 0].$$

By Proposition 3.5 we have that u_+ and v_- are the extremal constant sign solutions of (1.1). Therefore

$$K_{\Phi_\lambda^+} = \{0, u_+\} \quad \text{and} \quad K_{\Phi_\lambda^-} = \{v_-, 0\}.$$

This proves (3.33).

Next, we are going to show that

$$\begin{aligned}
&u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right) \text{ and } v_- \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right) \text{ are local} \\
&\text{minimizers of } \Phi_\lambda.
\end{aligned} \tag{3.36}$$

We easily verify that the functional Φ_λ^+ is coercive and sequentially weakly lower semicontinuous. Then, by the Weierstrass theorem, there exists $\hat{u} \in W^{1,p}(\Omega)$ such

that

$$\Phi_\lambda^+\left(\hat{u}\right) = \inf\left\{\Phi_\lambda^+(u) : u \in W^{1,p}(\Omega)\right\}.$$

Applying the same arguments as in the proof of Proposition 3.3, we can show that $\Phi_\lambda^+\left(\hat{u}\right) < 0 = \Phi_\lambda^+(0)$ which implies $\hat{u} \neq 0$. Hence, because of (3.33), $\hat{u} = u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. Since $\Phi_\lambda\big|_{C^1(\overline{\Omega})_+} = \Phi_\lambda^+\big|_{C^1(\overline{\Omega})_+}$ we know that $u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ is a local $C^1(\overline{\Omega})$-minimizer of Φ_λ and thanks to Proposition 2.6 it follows that u_+ is a local $W^{1,p}(\Omega)$-minimizer of Φ_λ. The assertion for Φ_λ^- can be shown using similar arguments. This proves (3.36).

Now, we may assume, without loss of generality, that $\Phi_\lambda(v_-) \leq \Phi_\lambda(u_+)$ and that u_+ is an isolated element of K_{Φ_λ}, otherwise we would have a whole sequence of distinct nontrivial solutions of (1.1). Then, we can find a number $\rho \in (0, 1)$ such that $\|v_- - u_+\|_{1,p} > \rho$ and

$$\Phi_\lambda\left(v_-\right) \leq \Phi_\lambda\left(u_+\right) < \inf\left[\Phi_\lambda(u) : \|u - u_+\|_{1,p} = \rho\right] = m_\rho^\lambda. \tag{3.37}$$

From Proposition 2.9 we have that Φ_λ satisfies the PS-condition because it is coercive. This fact along with (3.37) allow us the application of the mountain pass theorem stated in Theorem 2.2 which guarantees the existence of $y_0 \in W^{1,p}(\Omega)$ such that

$$y_0 \in K_{\Phi_\lambda} \quad \text{and} \quad m_\rho^\lambda \leq \Phi_\lambda(y_0). \tag{3.38}$$

Note that the first assertion in (3.38) combined with (3.33) and the definition of the truncations in (3.31), (3.32) implies that y_0 is a solution of problem (1.1). The second assertion in (3.38) along with (3.37) gives $y_0 \notin \{v_-, u_+\}$ and the nonlinear regularity theory yields that $y_0 \in C^1(\overline{\Omega})$. Since v_- and u_+ are the extremal constant sign solutions of (1.1) we know that $y_0 \in [v_-, u_+]\backslash\{v_-, u_+\}$ has changing sign provided $y_0 \neq 0$.

Moreover, since y_0 is of mountain pass type, we obtain, due to Theorem 2.2,

$$\Phi_\lambda(y_0) = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \Phi_\lambda(\gamma(t)), \tag{3.39}$$

where $\Gamma = \left\{\gamma \in C\left([0, 1], W^{1,p}(\Omega)\right) : \gamma(0) = v_-, \gamma(1) = u_+\right\}$. In order to complete the proof we have to show that y_0 is unequal zero which is satisfied if there exists a path $\gamma_* \in \Gamma$ such that (see (3.39))

$$\Phi_\lambda(\gamma_*(t)) \neq 0 \quad \text{for all} \quad t \in [0, 1].$$

Taking hypotheses (H3), (H6) into account, for given $\varepsilon_1, \varepsilon_2 > 0$, there exist numbers $\delta_1 = \delta_1(\varepsilon_1), \delta_2 = \delta_2(\varepsilon_2) > 0$ such that

$$\begin{aligned} |f(x, s)| &\leq \varepsilon_1 |s|^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and all } |s| \leq \delta_1, \\ |h(x, s)| &\leq \varepsilon_2 |s|^{q-1} \quad \text{for a.a. } x \in \partial\Omega \text{ and all } |s| \leq \delta_2, \end{aligned}$$

which implies that

$$\begin{aligned} F(x,s) &\le \frac{\varepsilon_1}{p}|s|^p \quad \text{for a.a. } x \in \Omega \text{ and all } |s| \le \delta_1, \\ H(x,s) &\le \frac{\varepsilon_2}{q}|s|^q \quad \text{for a.a. } x \in \partial\Omega \text{ and all } |s| \le \delta_2. \end{aligned} \tag{3.40}$$

Let $S_q = W^{1,q}(\Omega) \cap \partial B_1^{q,\partial\Omega}$ and $S_q^c = S_q \cap C^1(\overline{\Omega})$ be equipped with the relative $W^{1,p}(\Omega)$-topology and the relative $C^1(\overline{\Omega})$-topology, respectively. Recall that $\partial B_1^{q,\partial\Omega} = \left\{u \in L^q(\partial\Omega) : \|u\|_{q,\partial\Omega} = 1\right\}$ and

$$\hat{\Gamma}(q) = \left\{\hat{\gamma} \in C\left([-1,1], S_q\right) : \hat{\gamma}(-1) = -\hat{u}_1(q), \hat{\gamma}(1) = \hat{u}_1(q)\right\}.$$

Moreover, we consider the set of continuous paths

$$\hat{\Gamma}_c(q) = \left\{\hat{\gamma} \in C\left([-1,1], S_q^c\right) : \hat{\gamma}(-1) = -\hat{u}_1(q), \hat{\gamma}(1) = \hat{u}_1(q)\right\}.$$

Let $\delta := \min\{\delta_1, \delta_2\}$. From the variational characterization of the second eigenvalue $\hat{\lambda}_2(q)$ (see Proposition 2.8), we find $\hat{\gamma} \in \hat{\Gamma}(q)$ such that

$$\max_{-1\le t\le 1} \left\|\hat{\gamma}(t)\right\|_{1,q}^q < \hat{\lambda}_2(q) + \frac{\delta}{2}. \tag{3.41}$$

It is well known that S_q^c is dense in S_q. This implies the density of $\hat{\Gamma}_c(q)$ in $\hat{\Gamma}(q)$ (see, for example, Winkert [22, Proof of Theorem 3.1.16]). Therefore, for a given $\varepsilon > 0$, there exists $\hat{\gamma}_0 \in \hat{\Gamma}_c(q)$ such that

$$\max_{-1\le t\le 1} \left\|\hat{\gamma}(t) - \hat{\gamma}_0(t)\right\|_{1,q} < \varepsilon. \tag{3.42}$$

Selecting $\varepsilon \in \left(0, \left(\hat{\lambda}_2(q) + \delta\right)^{\frac{1}{q}} - \left(\hat{\lambda}_2(q) + \frac{\delta}{2}\right)^{\frac{1}{q}}\right)$ we derive from (3.41) and (3.42)

$$\begin{aligned} \left\|\hat{\gamma}_0(t)\right\|_{1,q} &\le \left\|\hat{\gamma}_0(t) - \hat{\gamma}(t)\right\|_{1,q} + \left\|\hat{\gamma}(t)\right\|_{1,q} \\ &< \varepsilon + \left(\hat{\lambda}_2(q) + \frac{\delta}{2}\right)^{\frac{1}{q}} \\ &< \left(\hat{\lambda}_2(q) + \delta\right)^{\frac{1}{q}} \quad \text{for all} \quad t \in [-1,1], \end{aligned}$$

which results in

$$\max_{-1\le t\le 1} \left\|\hat{\gamma}_0(t)\right\|_{1,q}^q < \hat{\lambda}_2(q) + \delta. \tag{3.43}$$

Recall that $u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ and $v_- \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. Then, since $\hat{\gamma}_0([-1,1]) \subseteq C^1(\overline{\Omega})$ is compact, there exists a number $\xi \in (0,1)$ such that

$$|\xi u(x)| \leq \delta \quad \text{for all} \quad x \in \overline{\Omega}, \ \text{for all} \quad u \in \hat{\gamma}_0([-1,1]), \tag{3.44}$$

and

$$\xi u \in [v_-, u_+] \quad \text{for all} \quad u \in \hat{\gamma}_0([-1,1]).$$

Due to (3.1) along with $\|u\|_{q,\partial\Omega} = 1$, (3.31), (3.32), (3.40), (3.43), (3.44), and $u \in \hat{\gamma}_0([-1,1])$, it follows

$$\begin{aligned}
\Phi_\lambda(\xi u) &= \int_\Omega G(\nabla(\xi u))dx + \frac{\chi}{p}\int_\Omega |\xi u|^p dx - \int_\Omega \Theta(x,\xi u)dx - \int_{\partial\Omega} \Theta_\lambda(x,\xi u)d\sigma \\
&\leq c_7\left(\xi^q \|\nabla u\|_{q,\Omega}^q + \xi^p \|\nabla u\|_{p,\Omega}^p\right) + \frac{\chi\xi^p}{p}\|u\|_{p,\Omega}^p \\
&\quad + \int_\Omega F(x,\xi u)dx - \frac{\lambda\xi^q}{q} + \int_{\partial\Omega} H(x,\xi u)d\sigma \\
&\leq c_7\left(\xi^q \|\nabla u\|_{q,\Omega}^q + \xi^q \|u\|_{q,\Omega}^q + \xi^p \|\nabla u\|_{p,\Omega}^p\right) - \xi^q c_7 \|u\|_{q,\Omega}^q \\
&\quad + \frac{\chi\xi^p}{p}\|u\|_{p,\Omega}^p + \frac{\varepsilon_1\xi^p}{p}\|u\|_{p,\Omega}^p - \frac{\lambda\xi^q}{q} + \frac{\varepsilon_2}{q}\xi^q \\
&\leq \xi^q\left[\frac{qc_7\left(\hat{\lambda}_2(q)+\delta\right) - \lambda + \varepsilon_2}{q}\right] - \xi^q c_7 \|u\|_{q,\Omega}^q \\
&\quad + \xi^p\left(c_7\|\nabla u\|_{p,\Omega}^p + \frac{\chi+\varepsilon_1}{p}\|u\|_{p,\Omega}^p\right).
\end{aligned} \tag{3.45}$$

Furthermore, since $\hat{\gamma}_0([-1,1]) \subseteq C^1(\overline{\Omega})$ is compact, we find a number $\xi^* > 0$ such that

$$\|u\|_{1,p}^p \leq \xi^* \quad \text{for all} \quad u \in \hat{\gamma}_0([-1,1]). \tag{3.46}$$

First, suppose that $q < p$. We choose $\varepsilon_1 \in (0,\infty)$, $\varepsilon_2 > 0$ and $\delta \in (0,\varepsilon_2)$ such that $qc_7\delta + \varepsilon_2 < \lambda - qc_7\hat{\lambda}_2(q)$. Taking (3.46) into account, (3.45) becomes

$$\Phi_\lambda(\xi u) \leq -\xi^q M_{12} + \xi^p M_{13} \quad \text{for some} \quad M_{12}, M_{13} > 0.$$

Since $q < p$, by choosing $\xi \in (0,1)$ small enough, we obtain

$$\Phi_\lambda(\xi u) < 0 \quad \text{for all} \quad u \in \hat{\gamma}_0([-1,1]). \tag{3.47}$$

If $q = p$, by applying (3.46) and again (3.43), estimate (3.45) becomes

$$\begin{aligned}\Phi_\lambda(\xi u) &\le \xi^p \left[\frac{pc_7\left(\hat{\lambda}_2(p)+\delta\right)-\lambda+\varepsilon_2}{p}\right] - \xi^p c_7 \|u\|_{p,\Omega}^p \\ &\quad + \xi^p \left(c_7\left(\|\nabla u\|_{p,\Omega}^p + \|u\|_{p,\Omega}^p\right) - c_7\|u\|_{p,\Omega}^p + \frac{\chi+\varepsilon_1}{p}\|u\|_{p,\Omega}^p\right) \quad (3.48)\\ &\le \xi^p \left[\frac{2pc_7\left(\hat{\lambda}_2(p)+\delta\right)-\lambda+\varepsilon_2}{p}\right] + \xi^p \frac{\chi+\varepsilon_1-2pc_7}{p}\xi^*.\end{aligned}$$

If $2pc_7 > \chi$, we choose $0 < \varepsilon_1 < 2pc_7 - \chi$ as well as $\varepsilon_2 > 0$ and $\delta \in (0, \varepsilon_2)$ such that $pc_7\delta + \varepsilon_2 < \lambda - pc_7\hat{\lambda}_2(p)$ which proves (3.47). If $2pc_7 = \chi$, then we select again $\varepsilon_2 > 0$ and $\delta \in (0, \varepsilon_2)$ such that $pc_7\delta + \varepsilon_2 < \lambda - pc_7\hat{\lambda}_2(p)$ for which (3.48) results in

$$\Phi_\lambda(\xi u) \le \xi^p \left[-M_{14} + \frac{\varepsilon_1}{p}\xi^*\right] \quad (3.49)$$

with some $M_{14} > 0$. Choosing $0 < \varepsilon_1 < \frac{pM_{14}}{\xi^*}$ proves (3.47) in this case, too.
Now, we set $\gamma_0 = \xi\hat{\gamma}_0$ which is a continuous path in $W^{1,p}(\Omega)$ connecting $-\xi\hat{u}_1(q)$ and $\xi\hat{u}_1(q)$ and which fulfills

$$\Phi_\lambda\big|_{\gamma_0} < 0. \quad (3.50)$$

Recall that, due to (3.33), $K_{\Phi_\lambda^+} = \{0, u_+\}$. Moreover, the proof of (3.36) shows that

$$\Phi_\lambda^+(u_+) = \inf_{u\in W^{1,p}(\Omega)} \Phi_\lambda^+(u) < 0 = \Phi_\lambda^+(0). \quad (3.51)$$

Now we may apply the second deformation theorem stated in Theorem 2.3 with $\varphi = \Phi_\lambda^+, a = \Phi_\lambda^+(u_+) < 0 = \Phi_\lambda^+(0) = b$ to find a continuous map $\hat{h} : [0,1] \times \left(\left(\Phi_\lambda^+\right)^0 \setminus\{0\}\right) \to \left(\Phi_\lambda^+\right)^0$ such that, because of (3.51) and (3.33),

$$\hat{h}\left(1, \left(\Phi_\lambda^+\right)^0 \setminus\{0\}\right) = \{u_+\} \quad (3.52)$$

and

$$\Phi_\lambda^+(\hat{h}(t,u)) \le \Phi_\lambda^+(u) \quad \text{for all} \quad t \in [0,1] \quad \text{and all} \quad u \in \left(\Phi_\lambda^+\right)^0 \setminus\{0\}. \quad (3.53)$$

Defining $\gamma_+(t) := \left(\hat{h}(t, \xi\hat{u}_1(q))\right)^+$ for all $t \in [0, 1]$, it is clear that γ_+ is a continuous path in $W^{1,p}(\Omega)$ satisfying

$$\gamma_+(0) = \left(\hat{h}(0, \xi\hat{u}_1(q))\right)^+ = \xi\hat{u}_1(q)$$

and, due to (3.52),

$$\gamma_+(1) = \left(\hat{h}(1, \xi\hat{u}_1(q)\right)^+ = u_+.$$

In addition, thanks to (3.53) and (3.50), one gets

$$\Phi_\lambda^+(\gamma_+(t)) \leq \Phi_\lambda^+(\xi\hat{u}_1(q)) < 0 \quad \text{for all} \quad t \in [0, 1]$$

implying $\Phi_\lambda^+\big|_{\gamma_+} < 0$. Moreover, since

$$\Phi_\lambda\big|_{W_+^p} = \Phi_\lambda^+\big|_{W_+^p} \quad \text{and} \quad \operatorname{im}\gamma_+ \subseteq W_+^p,$$

with $W_+^p = \{u \in W^{1,p}(\Omega) : u(x) \geq 0 \text{ a.e. in } \Omega\}$, we have

$$\Phi_\lambda\big|_{\gamma_+} < 0. \tag{3.54}$$

Following the same ideas we can construct a continuous path γ_- in $W^{1,p}(\Omega)$ which joins v_- and $-\xi\hat{u}_1(q)$ satisfying

$$\Phi_\lambda\big|_{\gamma_-} < 0. \tag{3.55}$$

The union of the curves γ_-, γ_0, and γ_+ forms a continuous path $\gamma_* \in \Gamma$ such that, because of (3.50), (3.54), and (3.55),

$$\Phi_\lambda\big|_{\gamma_*} < 0.$$

This implies that $y_0 \in C^1(\overline{\Omega}) \cap [v_-, u_+]$ is a nodal solution of (1.1). □

Combining the results in Propositions 3.3, 3.5, and 3.6 we have the following multiplicity result.

Theorem 3.7 *Let hypotheses $H(a)_1$ and (H1)–(H8) be satisfied and assume*

$$\lambda > \begin{cases} qc_7\hat{\lambda}_2(q) & \text{if } q < p, \\ 2pc_7\hat{\lambda}_2(p) & \text{if } q = p. \end{cases}$$

Then problem (1.1) has at least three nontrivial solutions $u_0 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$, $v_0 \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ and a nodal solution $y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega})$. Additionally, (1.1) has

a smallest nontrivial positive solution $u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ *and a greatest nontrivial negative solution* $v_- \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$.

Remark 3.8 As mentioned in the Introduction recall that the results in Theorem 3.7 recover those ones obtained in Winkert [23]. Indeed, if $q = p$ and $a(\xi) = \|\xi\|_{\mathbb{R}^N}^{p-2}\xi$ for all $\xi \in \mathbb{R}^N$ is the p-Laplacian, then $c_7 = \frac{1}{2p}$ and $2pc_7\hat{\lambda}_2(p) = \hat{\lambda}_2(p)$ being the second eigenvalue of the p-Laplacian with Steklov boundary condition. In this case problem (1.1) becomes

$$\begin{aligned} -\Delta_p u &= -\chi|u|^{p-2}u - f(x,u) && \text{in } \Omega, \\ \|\nabla u\|_{\mathbb{R}^N}^{p-2}\frac{\partial u}{\partial n} &= \lambda|u|^{p-2}u - h(x,u) && \text{on } \partial\Omega. \end{aligned}$$

with

$$0 < \chi \leq 1.$$

In contrast to [23] we have on the one hand a more general operator being possibly nonhomogeneous and on the other hand we do not need a sign-changing condition on f near the origin.

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Variational–hemivariational inequalities with small perturbations of nonhomogeneous Neumann boundary conditions

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ABSTRACT

In this paper variational–hemivariational inequalities with nonhomogeneous Neumann boundary conditions are investigated. Under an appropriate oscillating behavior of the nonlinear term, the existence of infinitely many solutions to this type of problems, even under small perturbations of nonhomogeneous Neumann boundary conditions, is established.

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1. Introduction

The aim of this paper is to investigate variational–hemivariational inequalities with a nonhomogeneous Neumann boundary condition. Precisely, let Ω be a non-empty, bounded, open subset of the Euclidean space $\mathbb{R}^N$, $N \geqslant 1$, with C^1-boundary $\partial\Omega$, let $p \in]N, +\infty[$, and let $q \in L^\infty(\Omega)$ satisfy $q \geqslant 0, q \not\equiv 0$. Our purpose is to study the following problem: Find $u \in K$ such that, for all $v \in K$,

$$\begin{aligned}&\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\big(v(x)-u(x)\big)\,dx + \int_\Omega q(x)|u(x)|^{p-2}u(x)\big(v(x)-u(x)\big)\,dx\\ &\quad+\int_\Omega \lambda\alpha(x)F^\circ\big(u(x);v(x)-u(x)\big)\,dx+\int_{\partial\Omega}\mu\beta(x)G^\circ\big(\gamma u(x);\gamma v(x)-\gamma u(x)\big)\,d\sigma \geqslant 0,\end{aligned}\tag{P}$$

where K is a closed convex subset of $W^{1,p}(\Omega)$ containing the constant functions, and $\alpha \in L^1(\Omega)$, $\beta \in L^1(\partial\Omega)$, with $\alpha(x) \geqslant 0$ for a.a. $x \in \Omega$, $\alpha \not\equiv 0$, $\beta(x) \geqslant 0$ for a.a. $x \in \partial\Omega$, and λ, μ are real parameters, with $\lambda > 0$ and $\mu \geqslant 0$. Here, F° and G° stand for Clarke's generalized directional derivatives of locally Lipschitz functions $F, G : \mathbb{R} \to \mathbb{R}$ given by $F(\xi) = \int_0^\xi f(t)\,dt$, $G(\xi) = \int_0^\xi g(t)\,dt$, $\xi \in \mathbb{R}$, with $f, g : \mathbb{R} \to \mathbb{R}$ locally essentially bounded functions, and $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ denotes the trace operator. If $\Omega =]a, b[\subseteq \mathbb{R}$ and $h : \{a, b\} \to \mathbb{R}$, then $\int_{\partial\Omega} h(x)\,d\sigma$ reads $h(b) + h(a)$, so problem (P) makes sense even for $N = 1$.

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A prototype of (P) for $K = W^{1,p}(\Omega)$ is the following boundary value problem with non-smooth potential and nonhomogeneous, non-smooth Neumann boundary condition

$$\begin{cases} \Delta_p u - q(x)|u|^{p-2}u \in \lambda\alpha(x)\partial F(u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} \in -\mu\beta(x)\partial G(\gamma u) & \text{on } \partial\Omega. \end{cases} \tag{N}$$

The main result of this paper is Theorem 3.1, which establishes, under an appropriate oscillating behavior of F and a suitable growth of G at infinity, the existence of a precise interval for the real parameter λ such that, provided μ is small enough, problem (P) admits infinitely many solutions. Some consequences and applications are also pointed out (see Theorems 3.6, 3.8 and Section 4). Just as an example, we illustrate the applicability of our approach by stating the following consequence of our results in the special case of ordinary differential equations.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative, locally essentially bounded function and set $F(\xi) = \int_0^\xi f(t)\,dt$ for all $\xi \in \mathbb{R}$. Assume that*

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0 \quad \textit{and} \quad \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = +\infty.$$

Then, for each non-negative, continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$g_\infty := \lim_{t\to+\infty} \frac{g(t)}{t} < +\infty,$$

and for every $\mu \in \left]0, \frac{1}{8g_\infty}\right[$, there is a sequence of pairwise distinct functions $\{u_n\} \subset W^{2,2}(]0,1[)$ such that for all $n \in \mathbb{N}$ one has

$$\begin{cases} -u_n''(x) + u_n(x) \in \left[f^-\big(u_n(x)\big),\, f^+\big(u_n(x)\big)\right] & \textit{for a.a. } x \in]0,1[, \\ u_n'(0) = \mu g\big(u_n(0)\big), \\ u_n'(1) = -\mu g\big(u_n(1)\big), \end{cases} \tag{ON}$$

where $f^-(t) = \lim_{\delta\to0^+} \operatorname{ess\,inf}_{|t-z|<\delta} f(z)$ and $f^+(t) = \lim_{\delta\to0^+} \operatorname{ess\,sup}_{|t-z|<\delta} f(z)$ for all $t \in \mathbb{R}$.

Clearly, if f is a continuous function, then Theorem 1.1 ensures the existence of infinitely many classical solutions to the Neumann boundary value problem (ON). It is worth noticing that our results are completely novel, even for continuous nonlinearities f and g, because of the presence of nonhomogeneous Neumann boundary condition (see (N)). We refer to [2] and [3], and the references therein, for smooth Neumann problems in the homogeneous case. Moreover, we also observe that our results and those of [11] are different since in [11] the Neumann boundary condition is homogeneous and the type of oscillating behavior at infinity required for f implies that f cannot be of constant sign, which is not necessary the case here. Regarding the existence of infinitely many solutions for non-smooth Neumann-type problems we also mention the paper of Candito [5] and the work of Kristály and Motreanu (see [9]) where in the second paper the authors don't require that $W^{1,p}(\Omega)$ is continuously embedded into $C^0(\overline{\Omega})$. Recently, Kristály and Moroşanu have described a new competition phenomena between oscillatory and pure power terms (cf. [8]) while existence results for variational–hemivariational inequalities of type (P) were established in [15] applying an abstract non-smooth critical point result given in [11].

2. Preliminaries

In this section we give a brief overview on some prerequisites on non-smooth analysis which are needed in the sequel. Let $(X, \|\cdot\|)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. A function $h : X \to \mathbb{R}$ is called locally Lipschitz continuous when to every $x \in X$ there correspond a neighborhood V_x of x and a constant $L_x \geqslant 0$ such that

$$\big|h(z) - h(w)\big| \leqslant L_x\|z - w\|, \quad \forall z, w \in V_x.$$

If $x, z \in X$, we write $h^\circ(x; z)$ for the generalized directional derivative of h at the point x along the direction z, i.e.,

$$h^\circ(x; z) := \limsup_{w\to x, t\to0^+} \frac{h(w + tz) - h(w)}{t}$$

(see [7, Chapter 2]). If $h_1, h_2 : X \to \mathbb{R}$ are locally Lipschitz functions, we have

$$(h_1 + h_2)^\circ(x, z) \leqslant h_1^\circ(x, z) + h_2^\circ(x, z), \quad \forall x, z \in X. \tag{2.1}$$

The generalized gradient of the function h at x, denoted by $\partial h(x)$, is the set

$$\partial h(x) := \left\{x^* \in X^* : \left\langle x^*, z\right\rangle \leqslant h^\circ(x; z),\ \forall z \in X\right\}.$$

We say that $x \in X$ is a (generalized) critical point of h when

$$h^\circ(x; z) \geqslant 0, \quad \forall z \in X,$$

that clearly means $0 \in \partial h(x)$ (see [6]).

When a non-smooth function $I : X \to]-\infty, +\infty]$ is expressed as a sum of a locally Lipschitz function, $h : X \to \mathbb{R}$, and a convex, proper and lower semicontinuous function, $j : X \to]-\infty, +\infty]$, that is $I := h + j$, a (generalized) critical point of I is every $u \in X$ such that

$$h^\circ(u; v - u) + j(v) - j(u) \geqslant 0,$$

for all $v \in X$ (see [13, Chapter 3] and [14]).

From now on, assume that X is a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous functional, $\Upsilon : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous functional, λ is a positive real parameter, $j : X \to]-\infty, +\infty]$ is a convex, proper and lower semicontinuous functional and $D(j)$ is the effective domain of j. Write

$$\Psi := \Upsilon - j \quad \text{and} \quad J_\lambda := \Phi - \lambda\Psi = (\Phi - \lambda\Upsilon) + \lambda j.$$

We also assume that Φ is coercive and

$$D(j) \cap \Phi^{-1}\left(]-\infty, r[\right) \neq \emptyset \tag{2.2}$$

for all $r > \inf_X \Phi$. Moreover, by (2.2) and provided $r > \inf_X \Phi$, we can define

$$\varphi(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{(\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v)) - \Psi(u)}{r - \Phi(u)},$$

and

$$\varphi^+ := \liminf_{r \to +\infty} \varphi(r), \qquad \varphi^- := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Assuming also that Φ and Υ are locally Lipschitz continuous functionals, in [4] it is proved the following result, which is a version of [11, Theorem 1.1].

Theorem 2.1. *Under the above assumptions on X, Φ and Ψ, one gets*:

(a) *If $\varphi^+ < +\infty$ then, for each $\lambda \in]0, \frac{1}{\varphi^+}[$, the following alternative holds*:
either
(a_1) *J_λ possesses a global minimum,*
or
(a_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of J_λ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$.*

(b) *If $\varphi^- < +\infty$ then, for each $\lambda \in]0, \frac{1}{\varphi^-}[$, the following alternative holds*:
either
(b_1) *there is a global minimum of Φ which is also a local minimum of J_λ,*
or
(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of J_λ, with $\lim_{n\to\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ.*

We recall that the previous theorem is a non-smooth version of Ricceri's variational principle (see [16]).

On the space $W^{1,p}(\Omega)$ we consider the norm

$$\|u\| := \left(\int_\Omega \left(\left|\nabla u(x)\right|^p + q(x)\left|u(x)\right|^p\right) dx\right)^{\frac{1}{p}},$$

which is equivalent to the usual one (see for instance [12, Section 1.1.15]). Set

$$c := \sup_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|}, \tag{2.3}$$

where $\|u\|_\infty := \max_{x \in \overline{\Omega}} |u(x)|$. From (2.3) we infer that $c^p \|q\|_1 \geqslant 1$. If Ω is convex, an explicit upper bound for the constant c in (2.3) is

$$c \leqslant 2^{\frac{p-1}{p}} \max\left\{ \frac{1}{\|q\|_1^{1/p}}, \frac{d}{N^{1/p}} \left(\frac{p-1}{p-N} |\Omega| \right)^{\frac{p-1}{p}} \frac{\|q\|_\infty}{\|q\|_1} \right\}, \tag{2.4}$$

where $|\Omega|$ denotes the Lebesgue measure of the set Ω and $d := \operatorname{diam}(\Omega)$ (see, e.g., [1, Remark 1]). Finally, we set

$$A = \liminf_{\xi \to +\infty} \frac{\max_{|t| \leqslant \xi} (-F(t))}{\xi^p}, \qquad B = \limsup_{\xi \to +\infty} \frac{-F(\xi)}{\xi^p},$$

and

$$\lambda_1 = \frac{\|q\|_1}{\|\alpha\|_1 p B}, \qquad \lambda_2 = \frac{1}{\|\alpha\|_1 p c^p A}. \tag{2.5}$$

3. Main results

Our main result is the following.

Theorem 3.1. *Let $\alpha \in L^1(\Omega)$ and $\beta \in L^1(\partial\Omega)$ be non-negative and non-zero functions. Let $f : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function and set $F(\xi) = \int_0^\xi f(t)\,dt$ for all $\xi \in \mathbb{R}$. Assume that*

$$A < \frac{1}{c^p \|q\|_1} B. \tag{3.1}$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, where λ_1, λ_2 are given by (2.5)*, for each locally essentially bounded function $g : \mathbb{R} \to \mathbb{R}$, whose potential $G(\xi) = \int_0^\xi g(t)\,dt$, $\xi \in \mathbb{R}$ satisfies*

$$G_\infty := \limsup_{\xi \to +\infty} \frac{\max_{|t| \leqslant \xi} (-G(t))}{\xi^p} < +\infty, \tag{3.2}$$

$$\liminf_{\xi \to +\infty} \big(-G(\xi)\big) > -\infty, \tag{3.3}$$

and for every $\mu \in [0, \delta[$, where

$$\delta = \delta_{g,\lambda} := \frac{1}{\beta^* p c^p G_\infty} \left(1 - \frac{\lambda}{\lambda_2} \right) \quad (\delta = +\infty \text{ if } G_\infty = 0),$$

with $\beta^* = \int_{\partial\Omega} \beta(x)\,d\sigma$, problem (P) *admits a sequence of weak solutions that is unbounded in $W^{1,p}(\Omega)$.*

Proof. Our aim is to apply Theorem 2.1. To this end, fix $\bar\lambda \in]\lambda_1, \lambda_2[$ and let g be a locally essentially bounded function satisfying our assumptions. Since $\bar\lambda < \lambda_2$, one has $\delta := \delta_{g,\bar\lambda} > 0$, so we can consider $0 \leqslant \bar\mu < \delta$. It follows that $\bar\lambda \|\alpha\|_1 p c^p A + \bar\mu \beta^* p c^p G_\infty < 1$, which implies

$$\bar\lambda < \frac{1}{\|\alpha\|_1 p c^p A + \frac{\bar\mu}{\bar\lambda} \beta^* p c^p G_\infty}. \tag{3.4}$$

Let X be the Sobolev space $W^{1,p}(\Omega)$ endowed with the norm $\|\cdot\|$. For any $u \in X$, set

$$\Phi(u) := \frac{1}{p}\|u\|^p, \qquad \Upsilon(u) := \int_\Omega \alpha(x)\big[-F\big(u(x)\big)\big]\,dx + \frac{\bar\mu}{\bar\lambda} \int_{\partial\Omega} \beta(x)\big[-G(\gamma u(x)\big]\,d\sigma,$$

$$j(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{otherwise,} \end{cases} \qquad \Psi(u) := \Upsilon(u) - j(u), \qquad J_\lambda(u) := \Phi(u) - \lambda\Psi(u).$$

Therefore,

$$J_{\bar\lambda}(u) = \frac{1}{p}\|u\|^p + \bar\lambda \int_\Omega \alpha(x) F\big(u(x)\big)\,dx + \bar\mu \int_{\partial\Omega} \beta(x) G\big(\gamma u(x)\big)\,d\sigma + \bar\lambda j(u) \quad \text{for all } u \in X. \tag{3.5}$$

Now, we claim that $\varphi^+ < +\infty$. Let $\{\rho_n\}$ be a real sequence such that $\lim_{n\to+\infty} \rho_n = +\infty$ and

$$\lim_{n \to +\infty} \frac{\max_{|t| \leqslant \rho_n} (-F(t))}{\rho_n^p} = A.$$

Denote $r_n = \frac{1}{p}(\frac{\rho_n}{c})^p$ and let $v \in \Phi^{-1}(]-\infty, r_n[)$. Then, taking into account that $\|v\|^p < pr_n$ and $\|v\|_\infty \leqslant c\|v\|$, one has $|v(x)| \leqslant \rho_n$ for every $x \in \Omega$. Therefore, it follows

$$\begin{aligned}\varphi(r_n) &\leqslant \frac{\sup_{\|w\|^p<pr_n}(\int_\Omega \alpha(x)[-F(w(x))]\,dx + \frac{\bar\mu}{\bar\lambda}\int_{\partial\Omega}\beta(x)[-G(\gamma w(x))]\,d\sigma - j(w))}{r_n}\\ &\leqslant \frac{\|\alpha\|_1 \max_{|t|\leqslant\rho_n}(-F(t)) + \frac{\bar\mu}{\bar\lambda}\beta^* \max_{|t|\leqslant\rho_n}(-G(t))}{r_n}\\ &= pc^p\|\alpha\|_1\frac{\max_{|t|\leqslant\rho_n}(-F(t))}{\rho_n^p} + pc^p\frac{\bar\mu}{\bar\lambda}\beta^*\frac{\max_{|t|\leqslant\rho_n}(-G(t))}{\rho_n^p}.\end{aligned}$$

Hence, $\varphi^+ \leqslant \limsup_{n\to+\infty}\varphi(r_n) \leqslant pc^p\|\alpha\|_1 A + pc^p\frac{\bar\mu}{\bar\lambda}\beta^* G_\infty$. From (3.1) and (3.2) we obtain

$$\varphi^+ < +\infty,$$

and our claim is proved. Moreover, taking into account (3.4), we get

$$\bar\lambda < \frac{1}{\varphi^+}.$$

Next, we show that the function $J_{\bar\lambda}$ in (3.5) is unbounded from below. Let $\{d_n\}$ be a real sequence such that $\lim_{n\to+\infty} d_n = +\infty$ and

$$\lim_{n\to+\infty}\frac{(-F(d_n))}{d_n^p} = B. \tag{3.6}$$

Set $w_n(x) = d_n$ for all $x \in \Omega$ and $n \in \mathbb{N}$. Clearly, $w_n \in K \subset W^{1,p}(\Omega)$ for each $n \in \mathbb{N}$. We see that

$$\|w_n\|^p = d_n^p\|q\|_1$$

and

$$\begin{aligned}\Phi(w_n) - \bar\lambda\Psi(w_n) &= \frac{\|w_n\|^p}{p} - \bar\lambda\Big(\int_\Omega \alpha(x)\big[-F\big(w_n(x)\big)\big]\,dx + \frac{\bar\mu}{\bar\lambda}\int_{\partial\Omega}\beta(x)\big[-G\big(\gamma w_n(x)\big)\big]\,d\sigma\Big) + \bar\lambda j(w_n)\\ &= \frac{d_n^p\|q\|_1}{p} - \bar\lambda\Big(\|\alpha\|_1\big(-F(d_n)\big) + \frac{\bar\mu}{\bar\lambda}\beta^*\big(-G(d_n)\big)\Big),\end{aligned}$$

thus

$$J_{\bar\lambda}(w_n) = \frac{d_n^p\|q\|_1}{p} - \bar\lambda\Big(\|\alpha\|_1\big(-F(d_n)\big) + \frac{\bar\mu}{\bar\lambda}\beta^*\big(-G(d_n)\big)\Big). \tag{3.7}$$

If $B < +\infty$, by (2.5) and since $\bar\lambda > \lambda_1$, we can take $\epsilon \in]0, B - \frac{\|q\|_1}{p\|\alpha\|_1\bar\lambda}[$. From (3.6) there exists ν_ϵ such that

$$-F(d_n) > (B-\epsilon)d_n^p, \quad \forall n > \nu_\epsilon.$$

Combining with (3.7), one has

$$J_{\bar\lambda}(w_n) < d_n^p\Big(\frac{\|q\|_1}{p} - \bar\lambda\|\alpha\|_1(B-\epsilon)\Big) - \bar\mu\beta^*\big(-G(d_n)\big).$$

Since $\frac{\|q\|_1}{p} - \bar\lambda\|\alpha\|_1(B-\epsilon) < 0$ and, as known from (3.3), $\{(-G(d_n))\}$ is bounded from below, it follows that $\lim_{n\to+\infty} J_{\bar\lambda}(w_n) = -\infty$. If $B = +\infty$, fix $M > \frac{\|q\|_1}{p\|\alpha\|_1\bar\lambda}$. Then from (3.6) there exists ν_M such that

$$\big(-F(d_n)\big) > Md_n^p, \quad \forall n > \nu_M.$$

Arguing as before, one obtains

$$J_{\bar\lambda}(w_n) < d_n^p\Big(\frac{\|q\|_1}{p} - \bar\lambda\|\alpha\|_1 M\Big) - \bar\mu\beta^*\big(-G(d_n)\big).$$

By the choice of M, we have $\lim_{n\to+\infty} J_{\bar\lambda}(w_n) = -\infty$, which completes the proof that $J_{\bar\lambda}$ is unbounded from below. Then, from part (a) of Theorem 2.1, we know that the function $J_{\bar\lambda}$ admits a sequence of critical points $\{\bar u_n\} \subset X$ such that $\|\bar u_n\| \to \infty$ as $n \to \infty$. The fact that $\bar u_n \in X$ is a critical point of $J_{\bar\lambda}$ reads as

$$(\Phi - \bar{\lambda}\Upsilon)^{\circ}(\bar{u}_n; v - \bar{u}_n) + \bar{\lambda} j(v) - \bar{\lambda} j(\bar{u}_n) \geqslant 0 \quad \text{for all } v \in X. \tag{3.8}$$

It remains to prove that $\bar{u}_n$ solves problem (P). From (3.8) it follows that $\bar{u}_n \in K$ and

$$(\Phi - \bar{\lambda}\Upsilon)^{\circ}(\bar{u}_n; v - \bar{u}_n) \geqslant 0 \quad \text{for all } v \in K. \tag{3.9}$$

By (3.9) and (2.1) we infer that

$$\Phi'(\bar{u}_n; v - \bar{u}_n) + \bar{\lambda}(-\Upsilon)^{\circ}(\bar{u}_n; v - \bar{u}_n) \geqslant 0 \quad \text{for all } v \in K$$

or, equivalently,

$$\begin{aligned} &\int_{\Omega} \left|\nabla \bar{u}_n(x)\right|^{p-2} \nabla \bar{u}_n(x) \cdot \nabla\big(v(x) - \bar{u}_n(x)\big)\, dx \\ &\quad + \int_{\Omega} q(x) \left|\bar{u}_n(x)\right|^{p-2} \bar{u}_n(x)\big(v(x) - \bar{u}_n(x)\big)\, dx + \bar{\lambda}(-\Upsilon)^{\circ}(\bar{u}_n, v - \bar{u}_n) \geqslant 0, \quad \forall v \in K. \end{aligned} \tag{3.10}$$

By using (2.1) and formula (2) on p. 77 in [7], we have

$$\bar{\lambda}(-\Upsilon)^{\circ}(\bar{u}_n; v - \bar{u}_n) \leqslant \bar{\lambda} \int_{\Omega} \alpha(x) F^{\circ}\big(\bar{u}_n(x); v(x) - \bar{u}_n(x)\big)\, dx + \bar{\mu} \int_{\partial\Omega} \beta(x) G^{\circ}\big(\gamma \bar{u}_n(x); \gamma v(x) - \gamma \bar{u}_n(x)\big)\, d\sigma.$$

Inserting this into (3.10) leads to

$$\begin{aligned} &\int_{\Omega} \left|\nabla \bar{u}_n(x)\right|^{p-2} \nabla \bar{u}_n(x) \cdot \nabla\big(v(x) - \bar{u}_n(x)\big)\, dx + \int_{\Omega} q(x) \left|\bar{u}_n(x)\right|^{p-2} \bar{u}_n(x)\big(v(x) - \bar{u}_n(x)\big)\, dx \\ &\quad + \bar{\lambda}\left[\int_{\Omega} \alpha(x) F^{\circ}\big(\bar{u}_n(x); v(x) - \bar{u}_n(x)\big)\, dx + \frac{\bar{\mu}}{\bar{\lambda}} \int_{\partial\Omega} \beta(x) G^{\circ}\big(\gamma \bar{u}_n(x); \gamma v(x) - \gamma \bar{u}_n(x)\big)\, d\sigma\right] \geqslant 0 \end{aligned}$$

for every $v \in K$, which completes the proof. □

The solutions obtained in Theorem 3.1 for problem (P) corresponding to the parameters $\bar{\lambda}$ and $\bar{\mu}$ are local minima of the functional $J_{\bar{\lambda}}$ in (3.5) which is associated to (P). The following corollary demonstrates that, if the functional $J_{\bar{\lambda}}$ satisfies the Palais–Smale condition, there are solutions to (P) which are not local minima of $J_{\bar{\lambda}}$.

Corollary 3.2. *Under the hypotheses of Theorem* 3.1*, assume in addition that*

$$\min\left\{F(t) - \frac{1}{s}(-F)^{\circ}(t; t), G(t) - \frac{1}{s}(-G)^{\circ}(t; t)\right\} \geqslant -c_1|t|^{\theta} - d_1 \quad \textit{for all } t \in \mathbb{R}, \tag{3.11}$$

with constants $1 \leqslant \theta < p < s$ *and* $c_1, d_1 \geqslant 0$*. Let* $u \in W^{1,p}(\Omega)$ *be a solution provided by Theorem* 3.1 *for* (P) *corresponding to* $\bar{\lambda} \in]\lambda_1, \lambda_2[$ *and* $0 \leqslant \bar{\mu} < \delta$ *, so* u *is a local minimum of the functional* $J_{\bar{\lambda}}$ *in* (3.5)*. If* u *is isolated, there exists another solution* $w \in W^{1,p}(\Omega)$ *for* (P) *corresponding to* $\bar{\lambda}$ *and* $\bar{\mu}$ *which is not a local minimum of* $J_{\bar{\lambda}}$*.*

Proof. Let us check that the functional $J_{\bar{\lambda}} : W^{1,p}(\Omega) \to]-\infty, +\infty]$ given in (3.5) satisfies the Palais–Smale condition in the sense of [13, p. 64]. This amounts to saying that whenever a sequence $\{u_n\} \subset K$ is such that $J(u_n)$ is bounded and

$$J_{\bar{\lambda}}^{\circ}(u_n; v - u_n) \geqslant -\varepsilon_n \|v - u_n\| \quad \text{for all } v \in K, \tag{3.12}$$

with $\varepsilon_n \to 0^+$, contains a convergent subsequence. Setting $v = 0$ in (3.12) and combining with the inequality $J_{\bar{\lambda}}(u_n) \leqslant M$, for a constant $M > 0$, yield

$$\begin{aligned} &\left(\frac{1}{p} - \frac{1}{s}\right)\|u_n\|^p + \bar{\lambda} \int_{\Omega} \alpha(x)\left[F\big(u_n(x)\big) - \frac{1}{s}(-F)^{\circ}(u_n; u_n)\right] dx \\ &\quad + \bar{\mu} \int_{\partial\Omega} \beta(x)\left[G\big(\gamma u_n(x)\big) - \frac{1}{s}(-G)^{\circ}(\gamma u_n; \gamma u_n)\right] d\sigma \leqslant M + \frac{\varepsilon_n}{s}\|u_n\|. \end{aligned}$$

Using hypothesis (3.11), it is straightforward to prove that the sequence $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. Furthermore, since $-\Delta_p$ on $W^{1,p}(\Omega)$ fulfills the (S_+) property, by handling (3.12) we find that $\{u_n\}$ contains a convergent subsequence, so the Palais–Smale condition for the functional $J_{\bar{\lambda}}$ is satisfied. We are thus in a position to apply [10, Theorem 4.2] to the functional $J_{\bar{\lambda}}$ from which we achieve the desired conclusion. □

Remark 3.3. Relation (2.4) is useful to verify inequality (3.1) and to estimate the numbers λ_2 and δ in Theorem 3.1 provided the bounded domain Ω is convex.

Remark 3.4. Actually, Theorem 3.1 ensures that the sequence $\{\bar{u}_n\}$ of solutions of problem (P) satisfies the following sharper inequality:

$$\int_\Omega \left|\nabla \bar{u}_n(x)\right|^{p-2}\nabla \bar{u}_n(x)\cdot\nabla\big(v(x)-\bar{u}_n(x)\big)\,dx+\int_\Omega q(x)\left|\bar{u}_n(x)\right|^{p-2}\bar{u}_n(x)\big(v(x)-\bar{u}_n(x)\big)\,dx$$
$$+\bar{\lambda}U^\circ(\bar{u}_n;v-\bar{u}_n)+\bar{\mu}V^\circ(\bar{u}_n;v-\bar{u}_n)\geqslant 0,\quad \forall v\in K,$$

where $U(u)=\int_\Omega \alpha(x)F(u(x))\,dx$ and $V(u)=\int_{\partial\Omega}\beta(x)G(\gamma u(x))\,d\sigma$ for all $u\in W^{1,p}(\Omega)$.

Remark 3.5. In Theorem 3.1 the function g may not have an oscillating behavior at infinity (see, for instance, Example 4.3 where $g(u)=\sqrt{|u|}$). On the other hand, g must satisfy (3.2), namely it must have a suitable growth at infinity. It is worth noticing that when (3.2) fails, that is,

$$\limsup_{\xi\to+\infty}\frac{\max_{|t|\leqslant\xi}(-G(t))}{\xi^p}=+\infty, \tag{3.13}$$

the existence of infinitely many solutions to (P) can be again guaranteed, provided that

$$G_\infty^-:=\liminf_{\xi\to+\infty}\frac{\max_{|t|\leqslant\xi}(-G(t))}{\xi^p}<+\infty \tag{3.14}$$

(for which g is then oscillating at infinity) and assuming that f, possibly even not oscillating at infinity, satisfies the following conditions:

$$B_+:=\limsup_{\xi\to+\infty}\frac{\max_{|t|\leqslant\xi}(-F(t))}{\xi^p}<+\infty,\qquad \liminf_{\xi\to+\infty}\big(-F(t)\big)>-\infty. \tag{3.15}$$

To be precise, the following result holds: Let α, β, f be as in the statement of Theorem 3.1 and assume that f satisfies (3.15). Then, for each $\lambda\in\,]0,\frac{1}{\|\alpha\|_1 pc^pB_+}[$, for each locally essentially bounded function $g:\mathbb{R}\to\mathbb{R}$ satisfying (3.13) and (3.14), and for each $\mu\in\,]0,\delta[$, where

$$\delta:=\delta_{g,\lambda}:=\frac{1}{\beta^* pc^pG_\infty^-}\big(1-\lambda\|\alpha\|_1 pc^pB_+\big)\quad\big(\delta=+\infty\text{ if }G_\infty^-=0\big),$$

the problem (P) admits a sequence of weak solutions which is unbounded in $W^{1,p}(\Omega)$.

Now we point out two significant special cases of Theorem 3.1.

Theorem 3.6. *Let $\alpha\in L^1(\Omega)$ and $\beta\in L^1(\partial\Omega)$ be non-negative and non-zero. Let $f:\mathbb{R}\to\mathbb{R}$ be a non-positive, locally essentially bounded function, and set $F(\xi)=\int_0^\xi f(t)\,dt$ for every $\xi\in\mathbb{R}$. Assume that*

$$\liminf_{\xi\to+\infty}\frac{-F(\xi)}{\xi^p}<\frac{1}{c^p\|q\|_1}\limsup_{\xi\to+\infty}\frac{-F(\xi)}{\xi^p}. \tag{3.16}$$

Then, for each $\lambda\in\,]\lambda_1,\lambda_2[$, where λ_1,λ_2 are given by (2.5)*, for each non-positive, locally essentially bounded function $g:\mathbb{R}\to\mathbb{R}$, whose potential $G(\xi)=\int_0^\xi g(t)\,dt$, $\xi\in\mathbb{R}$, satisfies*

$$G_\infty:=\limsup_{\xi\to+\infty}\frac{-G(\xi)}{\xi^p}<+\infty, \tag{3.17}$$

and for every $\mu\in[0,\delta[$, where

$$\delta=\delta_{g,\lambda}:=\frac{1}{(\int_{\partial\Omega}\beta(x)\,d\sigma)pc^pG_\infty}\left(1-\lambda\|\alpha\|_1 pc^p\liminf_{\xi\to+\infty}\frac{(-F(\xi))}{\xi^p}\right),$$

problem (P) *admits a sequence of weak solutions that is unbounded in $W^{1,p}(\Omega)$.*

Remark 3.7. In Theorem 3.6 the assumption (3.16) can be written

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p} < c^p\|q\|_1\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p},$$

as well as

$$]\lambda_1,\lambda_2[= \left]-\frac{\|q\|_1}{\|\alpha\|_1 p\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}},-\frac{1}{\|\alpha\|_1 pc^p\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}}\right[$$

and

$$\delta = -\frac{1}{(\int_{\partial\Omega}\beta(x)\,d\sigma)pc^p(\liminf_{\xi\to+\infty}\frac{G(\xi)}{\xi^p})}\left(1+\lambda\|\alpha\|_1 pc^p\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}\right).$$

Theorem 3.8. *Let $\alpha\in L^1(\Omega)$ and $\beta\in L^1(\partial\Omega)$ be non-negative and non-zero. Let $f:\mathbb{R}\to\mathbb{R}$ be a non-positive, locally essentially bounded function and set $F(\xi)=\int_0^\xi f(t)\,dt$ for every $\xi\in\mathbb{R}$. Assume that*

$$\liminf_{\xi\to+\infty}\frac{-F(\xi)}{\xi^p}=0 \quad \text{and} \quad \limsup_{\xi\to+\infty}\frac{-F(\xi)}{\xi^p}=+\infty.$$

Then, for each non-positive, locally essentially bounded function $g:\mathbb{R}\to\mathbb{R}$ such that

$$g_\infty := \lim_{\xi\to+\infty}\frac{-g(\xi)}{\xi^{p-1}} < +\infty,$$

and for every $\mu\in[0,\delta[$, where

$$\delta=\delta_g := \frac{1}{(\int_{\partial\Omega}\beta(x)\,d\sigma)c^p g_\infty},$$

there is an unbounded sequence $\{u_n\}\subset W^{1,p}(\Omega)$ such that

$$\int_\Omega \big|\nabla u_n(x)\big|^{p-2}\nabla u_n(x)\cdot\nabla\big(v(x)-u_n(x)\big)\,dx+\int_\Omega q(x)\big|u_n(x)\big|^{p-2}u_n(x)\big(v(x)-u_n(x)\big)\,dx$$
$$+\int_\Omega\alpha(x)F^\circ\big(u_n(x);\big(v(x)-u_n(x)\big)\big)\,dx+\mu\int_{\partial\Omega}\beta(x)G^\circ\big(\gamma u_n(x);\big(\gamma v(x)-\gamma u_n(x)\big)\big)\,d\sigma\geqslant 0$$

for all $v\in K$.

Remark 3.9. We explicitly observe that in Theorem 3.1 no symmetry assumption on the nonlinear term is done. On the other hand, the case $N\leqslant p$ cannot be studied by this method since the embedding of $W^{1,p}(\Omega)$ in $C^0(\overline{\Omega})$ fails and φ^+ cannot be upper estimated, without further assumptions on the nonlinear term.

Remark 3.10. An example of application of previous results is given in the next section (see Example 4.3).

If F oscillates at zero, we can give an analogous result as in Theorem 3.1. To this end, let

$$A=\liminf_{\xi\to0^+}\frac{\max_{|t|\leqslant\xi}(-F(t))}{\xi^p},\qquad B=\limsup_{\xi\to0^+}\frac{-F(\xi)}{\xi^p},$$

and

$$\lambda_1=\frac{\|q\|_1}{\|\alpha\|_1 pB},\qquad \lambda_2=\frac{1}{\|\alpha\|_1 pc^pA}. \tag{3.18}$$

Then, our result reads as follows.

Theorem 3.11. *Let $\alpha\in L^1(\Omega)$ and $\beta\in L^1(\partial\Omega)$ be non-negative and non-zero functions. Let $f:\mathbb{R}\to\mathbb{R}$ be a locally essentially bounded function and set $F(\xi)=\int_0^\xi f(t)\,dt$ for all $\xi\in\mathbb{R}$. Assume that*

$$A<\frac{1}{c^p\|q\|_1}B.$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, *where* λ_1, λ_2 *are given by* (3.18), *for each locally essentially bounded function* $g : \mathbb{R} \to \mathbb{R}$, *whose potential* $G(\xi) = \int_0^\xi g(t)\,dt$, $\xi \in \mathbb{R}$ *satisfies*

$$G_0 := \limsup_{\xi \to 0^+} \frac{\max_{|t| \leqslant \xi}(-G(t))}{\xi^p} < +\infty,$$

$$\liminf_{\xi \to 0^+} \big(-G(\xi)\big) \geqslant 0,$$

and for every $\mu \in [0, \delta[$, *where*

$$\delta = \delta_{g,\lambda} := \frac{1}{\beta^* p c^p G_0}\left(1 - \frac{\lambda}{\lambda_2}\right) \quad (\delta = +\infty \text{ if } G_0 = 0),$$

with $\beta^* = \int_{\partial\Omega} \beta(x)\,d\sigma$, *problem* (P) *admits a sequence of distinct weak solutions converging uniformly to zero.*

Proof. The proof can be done similarly as the proof of Theorem 3.1 by applying part (b) of Theorem 2.1 instead of part (a), thus obtaining the assertion. □

4. Applications and examples

Here we present an application of Theorem 3.1 to an ordinary differential problem with discontinuous nonlinearities.

Theorem 4.1. *Let* $\alpha \in L^1(]0, 1[)$ *be a non-negative and non-zero function and let* β_1, β_0 *be non-negative constants such that at least one of them is positive. Let* $f : \mathbb{R} \to \mathbb{R}$ *be a non-positive, locally essentially bounded function and set* $F(\xi) = \int_0^\xi f(t)\,dt$ *for every* $\xi \in \mathbb{R}$. *Assume that*

$$\liminf_{\xi \to +\infty} \frac{-F(\xi)}{\xi^2} < \frac{1}{2} \limsup_{\xi \to +\infty} \frac{-F(\xi)}{\xi^2}. \tag{4.1}$$

Then, for each $\lambda \in \left]\frac{1}{\|\alpha\|_1 2 \limsup_{\xi \to +\infty} \frac{-F(\xi)}{\xi^2}}, \frac{1}{\|\alpha\|_1 4 \liminf_{\xi \to +\infty} \frac{-F(\xi)}{\xi^2}}\right[$, *for each non-positive, continuous function* $g : \mathbb{R} \to \mathbb{R}$, *whose potential* $G(\xi) = \int_0^\xi g(t)\,dt$, $\xi \in \mathbb{R}$, *satisfies*

$$G_\infty := \limsup_{\xi \to +\infty} \frac{-G(\xi)}{\xi^2} < +\infty,$$

and for every $\mu \in [0, \delta[$, *where*

$$\delta = \delta_{g,\lambda} := \frac{1}{(\beta_1 + \beta_0) 4 G_\infty}\left(1 - \lambda \|\alpha\|_1 4 \liminf_{\xi \to +\infty} \frac{-F(\xi)}{\xi^p}\right) \quad (\delta = +\infty \text{ if } G_\infty = 0),$$

there is a sequence of pairwise distinct functions $\{u_n\} \subset W^{2,2}(]0, 1[)$ *such that for all* $n \in \mathbb{N}$ *one has*

$$\begin{cases} u_n''(x) - u_n(x) \in \big[\lambda\alpha(x) f^-\big(u_n(x)\big), \lambda\alpha(x) f^+\big(u_n(x)\big)\big] & \text{for a.a. } x \in]0, 1[, \\ u_n'(0) = \mu\beta_0 g\big(u_n(0)\big), \\ u_n'(1) = -\mu\beta_1 g\big(u_n(1)\big). \end{cases} \tag{N1}$$

Proof. The result is a consequence of Theorem 3.6. For the sake of clarity, we first point out three facts specific for the ordinary differential case that enable us to adapt the proof of Theorem 3.1 to this situation. The first one is the inequality

$$\begin{aligned} \int_{\partial\Omega} \beta(x)\big[-G\big(\gamma u(x)\big)\big]\,d\sigma &= \beta(1)\big[-G\big(u(1)\big)\big] + \beta(0)\big[-G\big(u(0)\big)\big] \\ &\leqslant \beta(1) \max_{|\xi| \leqslant \rho_n}\big[-G(\xi)\big] + \beta(0) \max_{|\xi| \leqslant \rho_n}\big[-G(\xi)\big] = \big(\beta(1) + \beta(0)\big) \max_{|\xi| \leqslant \rho_n}\big[-G(\xi)\big] \end{aligned}$$

for all $\|u\|^p < pr_n$, from which we derive

$$\varphi^+ \leqslant \limsup_{n \to +\infty} \varphi(r_n) \leqslant pc^p \|\alpha\|_1 A + pc^p \frac{\bar{\mu}}{\bar{\lambda}}\big(\beta(1) + \beta(0)\big) G_\infty.$$

The second one is the estimate

$$\int_{\partial\Omega}\beta(x)\big[-G\big(\gamma w_n(x)\big)\big]\,d\sigma=\big(\beta(1)+\beta(0)\big)\big[-G(d_n)\big]\geqslant\big(\beta(1)+\beta(0)\big)\liminf_{\xi\to+\infty}\big(-G(\xi)\big)\geqslant 0,$$

from which we deduce $\lim_{n\to+\infty} J_{\bar\lambda}(w_n)=-\infty$. The last one is

$$\begin{aligned}\bigg[\int_{\partial\Omega}\beta(x)G\big(\gamma\bar u_n(x);\big(\gamma v(x)-\gamma\bar u_n(x)\big)\big)\,d\sigma\bigg]^\circ&=\big[\beta(1)G\big(\bar u_n(1);v(1)-\bar u_n(1)\big)+\beta(0)G\big(\bar u_n(0);v(0)-\bar u_n(0)\big)\big]^\circ\\&\leqslant\beta(1)G^\circ\big(\bar u_n(1);v(1)-\bar u_n(1)\big)+\beta(0)G^\circ\big(\bar u_n(0);v(0)-\bar u_n(0)\big),\end{aligned}$$

from which it turns out

$$\begin{aligned}&\int_0^1|\overline{u}'_n(x)|^{p-2}\overline{u}'_n(x)\cdot\big(v'(x)-\overline{u}'_n(x)\big)\,dx+\int_0^1 q(x)|\bar u_n(x)|^{p-2}\bar u_n(x)\big(v(x)-\bar u_n(x)\big)\,dx\\&\quad+\bar\lambda\int_0^1\alpha(x)F^\circ\big(\bar u_n(x);v(x)-\bar u_n(x)\big)\,dx+\bar\mu\big[\beta(1)G^\circ\big(\bar u_n(1);v(1)-\bar u_n(1)\big)+\beta(0)G^\circ\big(\bar u_n(0);v(0)-\bar u_n(0)\big)\big]\geqslant 0,\\&\forall v\in K.\end{aligned}$$

The proof of Theorem 4.1 is carried out as follows. Fix $\bar\lambda$ and $\bar\mu$ as in the conclusion of Theorem 4.1. We may apply Theorem 3.6 (see also Remark 3.4) by choosing $\Omega=]0,1[$, $p=2$, $q\equiv 1$, $K=W^{1,2}(]0,1[)$, and noticing that hypothesis (4.1) in conjunction with (2.4) implies that (3.16) holds true. Then there exists an unbounded sequence $\{\bar u_n\}\subset W^{1,2}(]0,1[)$ such that

$$\begin{aligned}&\int_0^1\overline{u}'_n(x)v'(x)\,dx+\int_0^1\bar u_n(x)v(x)\,dx+\bar\lambda U^\circ(\bar u_n;v)+\bar\mu\big[\beta_1G^\circ\big(\bar u_n(1);v(1)\big)+\beta_0G^\circ\big(\bar u_n(0);v(0)\big)\big]\geqslant 0,\\&\forall v\in W^{1,2}\big(]0,1[\big),\end{aligned}$$

where $\beta_0=\beta(0)$ and $\beta_1=\beta(1)$, while the function U was introduced in Remark 3.4. Setting

$$T_n(v)=-\bigg[\int_0^1\overline{u}'_n(x)v'(x)\,dx+\int_0^1\bar u_n(x)v(x)\,dx\bigg]-\bar\mu\big[\beta_1g\big(\bar u_n(1)\big)v(1)+\beta_0g\big(\bar u_n(0)\big)v(0)\big]$$

for all $v\in W^{1,2}(]0,1[)$, we see that T_n is linear and continuous on $W^{1,2}(]0,1[)$, and $T_n\in\bar\lambda\partial U(\bar u_n)$. Taking into account that $W^{1,2}(]0,1[)$ is continuously and densely embedded in $L^2(]0,1[)$, from [6, Theorem 2.2] we know that there is $h_n\in L^2(]0,1[)$ such that

$$-\bigg[\int_0^1\overline{u}'_n(x)v'(x)\,dx+\int_0^1\bar u_n(x)v(x)\,dx\bigg]-\bar\mu\big[\beta_1g\big(\bar u_n(1)\big)v(1)+\beta_0g\big(\bar u_n(0)\big)v(0)\big]=\int_0^1h_n(x)v(x)\,dx$$

for all $v\in W^{1,2}(]0,1[)$. This ensures that $\bar u_n$ is the unique solution of the problem

$$\begin{cases}u''-u=h_n(x)\quad\text{in }]0,1[,\\u'(0)=-\bar\mu\beta_0g\big(u(0)\big),\\u'(1)=\bar\mu\beta_1g\big(u(1)\big)\end{cases}$$

and, in addition, $\bar u_n\in W^{2,2}(]0,1[)$. Moreover, since $T_n\in\bar\lambda\partial U(\bar u_n)$, we deduce through [6, Corollary, p. 111] that

$$h_n(x)\in\big[\bar\lambda\alpha(x)f^-\big(\bar u_n(x)\big),\bar\lambda\alpha(x)f^+\big(\bar u_n(x)\big)\big]\quad\text{for a.a. }x\in]0,1[.$$

Hence the conclusion regarding problem (N1) is obtained with $u_n=\bar u_n$. □

Remark 4.2. Theorem 1.1 in the Introduction is a direct consequence of Theorem 4.1.

Example 4.3. Set

$$a_n:=\frac{2n!(n+2)!-1}{4(n+1)!},\qquad b_n:=\frac{2n!(n+2)!+1}{4(n+1)!}$$

for every $n \in \mathbb{N}$ and define the non-negative (discontinuous) function $f_p : \mathbb{R} \to \mathbb{R}$ by

$$f_p(t) = \begin{cases} 2(n+1)![n^{p-1}(n+1)!^p - (n-1)^{p-1}n!^p] & \text{if } t \in \bigcup_{n\in\mathbb{N}}]a_n, b_n[, \\ 0 & \text{otherwise.} \end{cases}$$

Denoting $F_p(\xi) = \int_0^\xi f_p(t)\,dt$ for every $\xi \in \mathbb{R}$, a simple computation shows that $\liminf_{\xi\to+\infty} \frac{F_p(\xi)}{\xi^p} = 0$ and $\limsup_{\xi\to+\infty} \frac{F_p(\xi)}{\xi^p} = +\infty$. Owing to Theorem 3.8, there is a sequence of pairwise distinct functions $\{u_n\} \subset W^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u_n(x)|^{p-2}\nabla u_n(x)\cdot\nabla\big(v(x)-u_n(x)\big)\,dx + \int_\Omega |u_n(x)|^{p-2}u_n(x)\big(v(x)-u_n(x)\big)\,dx + \int_\Omega (-F)^\circ\big(u_n(x); v(x)-u_n(x)\big)\,dx + \int_{\partial\Omega}\Big[-\big|\gamma u_n(x)\big|^{\frac{p-1}{2}}\Big]\big(\gamma v(x)-\gamma u_n(x)\big)\,d\sigma \geqslant 0, \quad \forall v \in K.$$

In particular, Theorem 4.1 ensures that there is a sequence of pairwise distinct functions $\{u_n\} \subset W^{2,2}(]0,1[)$ such that for all $n \in \mathbb{N}$ there holds

$$\begin{cases} -u_n''(x) + u_n(x) \in \big[f_2^-\big(u(x)\big), f_2^+\big(u(x)\big)\big] & \text{for a.a. } x \in]0,1[, \\ u_n'(0) = -\sqrt{|u_n(0)|}, \\ u_n'(1) = \sqrt{|u_n(1)|}. \end{cases}$$

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MULTIPLICITY RESULTS TO A CLASS OF VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

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Abstract. This paper deals with variational-hemivariational inequalities involving the p-Laplace operator and a nonlinear Neumann boundary condition. Based on an abstract critical point result, which is developed at the beginning of the paper, it is shown the existence of at least three solutions to such inequalities whereby the cases $p > N$ and $p \leq N$ are treated separately. The applicability of these results is emphasized with suitable examples.

1. Introduction

Let Ω be a non-empty, bounded, open subset of the real Euclidian space $\mathbb{R}^N$, $N \geq 1$, with C^1-boundary $\Gamma := \partial\Omega$ and let $q \in L^\infty(\Omega)$ satisfying $q \geq 0$, $q \not\equiv 0$. We consider the following problem: Find $u \in K$ such that, for all $v \in K$

$$\begin{aligned}(1.1)\qquad &\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla(v(x)-u(x))\,dx\\ &\quad+\int_\Omega q(x)|u(x)|^{p-2}u(x)(v(x)-u(x))\,dx\\ &\quad+\int_\Omega \lambda\alpha(x)F^\circ(u(x);v(x)-u(x))\,dx\\ &\quad+\int_\Gamma \mu\beta(x)G^\circ(\gamma u(x);\gamma v(x)-\gamma u(x))\,d\sigma\geq 0,\end{aligned}$$

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where K is a closed convex subset of the usual Sobolev space $W^{1,p}(\Omega), 1 < p < \infty$, containing the zero function while $\alpha \in L^1(\Omega)$, $\beta \in L^1(\Gamma)$, fulfill $\alpha(x) \geq 0$ for almost all $x \in \Omega$, $\alpha \not\equiv 0$ and $\beta(x) \geq 0$ for almost all $x \in \Gamma$. The values λ and μ are real parameters with $\lambda > 0$ and $\mu \geq 0$ specified later. By F° and G° we denote Clarke's generalized directional derivatives of the locally Lipschitz continuous functions $F, G\colon \mathbb{R} \to \mathbb{R}$ given in the form

$$F(\xi) = \int_0^\xi f(t)\,dt, \qquad G(\xi) = \int_0^\xi g(t)\,dt,$$

with locally essentially bounded functions $f, g\colon \mathbb{R} \to \mathbb{R}$. As usual, we denote by $\gamma\colon W^{1,p}(\Omega) \to L^p(\Gamma)$ the well-known trace operator being linear and compact.

The aim of this paper is to provide multiplicity results to inequality (1.1). We present different existence theorems showing the existence of at least three distinct solutions of (1.1) provided the number λ belongs to a specific interval and the parameter μ is sufficiently small. Since $C(\overline{\Omega})$ is compactly embedded into $W^{1,p}(\Omega)$ when $p > N$, the various cases $p \leq N$ and $p > N$ are discussed separately with different assumptions on the data f and g. Our main results are stated in Theorem 3.1 ($p > N$, see Section 3) and Theorem 4.1 ($p \leq N$, see Section 4). The main idea in the proofs is the usage of an appropriate abstract three-critical-point-result for non-smooth functionals which is proved in Section 2 based on results in [8].

Existence and multiplicity results for variational-hemivariational inequalities have been obtained under different structure and regularity conditions on the nonlinear functions by various authors. We refer, for example, to [4], [23], [24], [28] and [36] as well as the references therein. It is clear that problem (1.1) becomes a hemivariational inequality if K coincides with the whole space $W^{1,p}(\Omega)$. Such inequalities have been handled for example in [2], [3], [13], [15], [29], [34] and [35]. In the context of infinitely many solutions to (1.1) we refer to the recent results stated in [9] and [41] for $p > N$ while existence results to (1.1) via the method of sub- and supersolution can be found in [14] and [16].

Let us comment on some relevant special cases of (1.1).

(A) If $K = W^{1,p}(\Omega)$ and F, G are smooth, problem (1.1) reduces to

$$\begin{aligned}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,dx + \int_\Omega q(x)|u(x)|^{p-2}u(x)\varphi(x)\,dx \\ + \int_\Omega \lambda\alpha(x)F'(u(x))\varphi(x)\,dx + \int_\Gamma \mu\beta(x)G'(\gamma u(x))\gamma\varphi(x)\,d\sigma = 0,\end{aligned}$$

for all $\varphi \in W^{1,p}(\Omega)$ which means that $u \in W^{1,p}(\Omega)$ is the weak solution of the nonlinear boundary value problem

$$\begin{aligned} -\Delta_p u(x) + q(x)|u(x)|^{p-2}u(x) + \lambda\alpha(x)F'(u(x)) &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}(x) + \mu\beta(x)G'(\gamma u(x)) &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{1.2}$$

where $(\partial u/\partial \nu)(x) = |\nabla u|^{p-2}(\partial u/\partial n)(x)$ with $(\partial u/\partial n)(x)$ being the outer normal derivative of u at $x \in \Gamma$. Regarding existence and multiplicity of solutions to (1.2) we point out (without guarantee of completeness) the papers in [20]–[22], [30], [32], [37], [46], [48], and the references therein. Referring to homogeneous Neumann problems, the existence of at least three solutions in case $p > N$ was shown with different methods for example in [1], [5] and [6] (see also [7] for infinitely many solutions) while the more complicated case $p \leq N$ was recently studied in [19].

(B) In case $f = g = 0$, (1.1) is a classical variational inequality of the form

$$\begin{aligned} u \in K: \quad & \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla(v(x) - u(x))\, dx \\ & + \int_\Omega q(x)|u(x)|^{p-2}u(x)(v(x) - u(x))\, dx \geq 0 \end{aligned}$$

for all $v \in K$ whose treatment is well-known (see for example the monograph of Kinderlehrer in [25]).

(C) As mentioned above, problem (1.1) reduces to a so-called hemivariational inequality provided $K = W^{1,p}(\Omega)$. This contains as a special case the subsequent elliptic inclusion

$$\begin{aligned} -\Delta_p u(x) + q(x)|u(x)|^{p-2}u(x) + \lambda\alpha(x)\partial F(u(x)) &\ni 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}(x) + \mu\beta(x)\partial G(\gamma u(x)) &\ni 0 \quad \text{on } \Gamma, \end{aligned} \tag{1.3}$$

whereby the multivalued functions ∂F, ∂G stand for Clarke's generalized gradient (see Section 2 for more details). Concerning multiple solutions such inclusions have been studied in [31] and [47]. Regarding the existence of infinitely many solutions for homogeneous problems of type (1.3) we also mention the paper of Candito [11] and the work of Kristály–Motreanu (see [27]) where in the second paper the authors do not require that $W^{1,p}(\Omega)$ is continuously embedded into $C(\overline{\Omega})$. Likewise, we draw attention to a paper of Kristály–Moroşanu in which a new competition phenomena between oscillatory and pure power terms has been described (cf. [26]).

It should be noted that our variational-hemivariational inequality is equivalent to the multi-valued variational inequality

$$(1.4)\quad u \in K: \quad \begin{cases} \eta(x) \in \partial F(u(x)) \quad \text{a.e. in } \Omega,\ \xi(x) \in \partial G(\gamma u(x)) \text{ a.e. in } \Gamma, \\ \displaystyle\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla(v(x)-u(x))\,dx \\ \quad \displaystyle + \int_\Omega q(x)|u(x)|^{p-2}u(x)(v(x)-u(x))\,dx \\ \quad \displaystyle + \int_\Omega \lambda\alpha(x)\eta(x)(v(x)-u(x))\,dx \\ \quad \displaystyle + \int_\Gamma \mu\beta(x)\xi(x)(\gamma v(x)-\gamma u(x))\,d\sigma \geq 0, \quad \text{for all } v \in K, \end{cases}$$

provided the elements of $\partial F, \partial G$ fulfill a suitable growth condition and K has lattice structure, which means, if $u, v \in K$, then $\max\{u,v\}, \min\{u,v\} \in K$. In other words, u is a solution of (1.1) if and only if u is a solution of (1.4). This interesting result was recently published in [12].

We also would like to mention the recent work in [44] and references therein concerning three critical points theorems involving smooth functionals. For more information about (variational-)hemivariational inequalities we refer the reader to the monographs in [42] and [43].

2. Preliminaries

Let us recall some basic facts on non-smooth analysis which we will need in later considerations. Let $(X, \|\cdot\|)$ be a real Banach space and denote by X^* its dual space while the duality pairing between X and X^* is denoted by $\langle\cdot,\cdot\rangle$. A function $f\colon X \to \mathbb{R}$ is said to be locally Lipschitz continuous if for every $x \in X$ there exist a neighborhood U_x of x and a constant $L_x \geq 0$ such that

$$|f(y)-f(z)| \leq L_x\|y-z\|, \quad \text{for all } y,z \in U_x.$$

The term $f^\circ(x;y)$, $x,y \in X$ stands for the generalized directional derivative of f at the point x along the direction y which is given by

$$f^\circ(x;y) := \limsup_{z\to x,\, t\to 0^+} \frac{f(z+ty)-f(z)}{t},$$

(see [18, Chapter 2]). Let $f_1, f_2\colon X \to \mathbb{R}$ be locally Lipschitz continuous functions. Then we have

$$(2.1)\qquad (f_1+f_2)^\circ(x;y) \leq f_1^\circ(x;y) + f_2^\circ(x;y), \quad \text{for all } x,y \in X.$$

The generalized gradient of a locally Lipschitz continuous function f at x, denoted by $\partial f(x)$, is the set

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y\rangle \leq f^\circ(x;y), \text{ for all } y \in X\}.$$

An element $x \in X$ is a (generalized) critical point of f if it satisfies the condition

$$f^\circ(x; y) \geq 0, \quad \text{for all } y \in X,$$

or equivalently, $0 \in \partial f(x)$ (see [17]).

Let $I: X \to]-\infty, +\infty]$ be a non-smooth function represented as a sum of a locally Lipschitz continuous function $f: X \to \mathbb{R}$ and a convex, proper and lower semicontinuous function $j: X \to]-\infty, +\infty]$, that is $I := f + j$. An element $u \in X$ is a (generalized) critical point of I if

$$f^\circ(u; v - u) + j(v) - j(u) \geq 0, \quad \text{for all } v \in X$$

is satisfied (see [39, Chapter 3] and [40]).

Now, let us give the assumptions concerning our first result. For a reflexive Banach space X, the functional $\Phi: X \to \mathbb{R}$ is assumed to be sequentially weakly lower semicontinuous and coercive while $\Upsilon: X \to \mathbb{R}$ is supposed to be sequentially weakly upper semicontinuous. By λ we denote a positive real parameter, $j: X \to]-\infty, +\infty]$ is a convex, proper and lower semicontinuous functional and $D(j)$ stands for the effective domain of j. Then we define

$$\Psi := \Upsilon - j \quad \text{and} \quad J_\lambda := \Phi - \lambda\Psi = (\Phi - \lambda\Upsilon) + \lambda j := \Theta_\lambda + \lambda j, \tag{2.2}$$

while it is supposed that

$$D(j) \cap \Phi^{-1}(]-\infty, r[) \neq \emptyset, \quad \text{for all } r > \inf_X \Phi.$$

Now we define

$$\varphi_1(r) = \inf_{y \in \Phi^{-1}(]-\infty, r[)} \frac{\left(\sup\limits_{x \in \Phi^{-1}(]-\infty, r[)} \Psi(x)\right) - \Psi(y)}{r - \Phi(y)}, \quad \text{for all } r > \inf_X \Phi,$$

respectively,

$$\varphi_2(r) = \sup_{y \in \Phi^{-1}(]r, +\infty[)} \frac{\Psi(y) - \left(\sup\limits_{x \in \Phi^{-1}(]-\infty, r])} \Psi(x)\right)}{\Phi(y) - r}, \quad \text{for all } r < \sup_X \Phi.$$

We have the following result.

THEOREM 2.1. *Assume that there is* $r \in \left]\inf\limits_X \Phi, \sup\limits_X \Phi\right[$ *such that* $\varphi_1(r) < \varphi_2(r)$. *Further suppose that the functional* J_λ *is bounded from below and satisfies the* (PS)*-condition for each* $\lambda \in \Lambda :=]1/\varphi_2(r), 1/\varphi_1(r)[$. *Then, for each* $\lambda \in \Lambda$, J_λ *has three distinct critical points.*

PROOF. First, we observe that, thanks to [40, Corollary 1.3], J_λ is coercive. Now, we want to show that J_λ has a local minima $u_1 \in \Phi^{-1}(]-\infty, r[)$ and a local minima $u_2 \in \Phi^{-1}(]r, +\infty[)$. Let $\lambda \in \Lambda$ be fixed. We are going to show that there is $u_1 \in D(j) \cap \Phi^{-1}(]-\infty, r[)$ such that $J_\lambda(u_1) \leq J_\lambda(u)$ for all $u \in \Phi^{-1}(]-\infty, r[)$.

Taking into account $1/\lambda > \varphi_1(r)$, we find $\overline{u} \in D(j)$ such that $\Phi(\overline{u}) < r$ and $J_\lambda(\overline{u}) = \Phi(\overline{u}) - \lambda\Psi(\overline{u}) < r - \lambda \sup\limits_{\Phi(x)<r} \Psi(x)$. Putting $L := (r - \Phi(\overline{u}))/\lambda + \Psi(\overline{u})$ yields

$$\sup_{\Phi(x)<r} \Psi(x) < L. \tag{2.3}$$

Let us take $\Psi_L(u) = \min\{\Psi(u), L\}$. As j is sequentially weakly lower semicontinuous (see [10, Corollary III.8]), we can easily prove that Ψ_L is sequentially weakly upper semicontinuous. We take $J = \Phi - \lambda\Psi_L$ and note that J is sequentially weakly lower semicontinuous and coercive which guarantees that its global minimum, namely u_0, exists (see [45, Theorem 1.2]). If $J(u_0) = J(\overline{u})$, we put $u_1 = \overline{u}$, so $u_1 \in \Phi^{-1}(]-\infty, r[)$ is a local minima of J_λ. Let us consider the case $J(u_0) < J(\overline{u})$. Then, we obtain

$$\Phi(u_0) - \lambda\Psi_L(u_0) < \Phi(\overline{u}) - \lambda\Psi_L(\overline{u}),$$

thus

$$\Phi(u_0) < \lambda\Psi_L(u_0) + \Phi(\overline{u}) - \lambda\Psi_L(\overline{u}) \leq \lambda L + \Phi(\overline{u}) - \lambda\Psi(\overline{u}) = r.$$

Thanks to (2.3) we see at once that $\Psi(u_0) < L$. Hence, it follows

$$\Phi(u_0) - \lambda\Psi(u_0) = \Phi(u_0) - \lambda\Psi_L(u_0) \leq \Phi(u) - \lambda\Psi_L(u), \quad \text{for all } u \in X. \tag{2.4}$$

With the aid of (2.3), (2.4) results in

$$\Phi(u_0) - \lambda\Psi(u_0) \leq \Phi(u) - \lambda\Psi(u), \quad \text{for all } u \in \Phi^{-1}(]-\infty, r[).$$

Taking $u_1 = u_0$ proves the other case and hence, $u_1 \in \Phi^{-1}(]-\infty, r[)$ is a local minima of J_λ.

Now we prove the existence of a local minima $u_2 \in \Phi^{-1}(]r, \infty[)$. Since $\lambda > 1/\varphi_2(r)$, there exists $\overline{v} \in D(j)$ such that $\Phi(\overline{v}) > r$ and

$$\Phi(\overline{v}) - \lambda\Psi(\overline{v}) < r - \lambda \sup_{\Phi(x)\leq r} \Psi(x). \tag{2.5}$$

Let us introduce a functional $\Phi_r \colon X \to \mathbb{R}$ defined by $\Phi_r(x) := \max\{\Phi(x), r\}$. Clearly, Φ_r is sequentially weakly lower semicontinuous and coercive. Then, $J := \Phi_r - \lambda\Psi$ has the same properties, so there exists a global minimum v_0 of J, that is,

$$\Phi_r(v_0) - \lambda\Psi(v_0) \leq \Phi_r(x) - \lambda\Psi(x), \quad \text{for all } x \in X. \tag{2.6}$$

Let us show that $v_0 \in \Phi^{-1}(]r, \infty[)$. We argue indirectly and assume that $v_0 \in \Phi^{-1}(]-\infty, r])$. Then, due to (2.6) with the special choice $x = \overline{v}$ and the fact that $\Phi(\overline{u}) > r$, it follows

$$r - \lambda\Psi(v_0) \leq \Phi(\overline{v}) - \lambda\Psi(\overline{v}).$$

Applying (2.5) yields

$$r - \lambda\Psi(v_0) \leq \Phi(\overline{v}) - \lambda\Psi(\overline{v}) < r - \lambda\Psi(v_0),$$

which is a contradiction. Hence, $u_2 := v_0 \in \Phi^{-1}(]r, \infty[)$ is a local minima of J_λ.

Now, we may apply Corollary 2.1 in [35] to obtain the existence of at least three critical points of J_λ. This completes the proof. □

Now, we present a variant of Theorem 2.1 where the (PS)-condition and the boundedness from below of J_λ are not required. To this end, let K be a non-empty closed convex subset of X containing the zero of X and let $j\colon X \to]-\infty, +\infty]$ be defined as

$$j(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, j is convex, proper and lower semicontinuous. So, we can consider the functionals as defined in (2.2). Assume also that

(1) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
(2) Φ is convex.

Moreover, for fixed $\lambda > 0$, suppose that

(3) for all $x_1, x_2 \in X$ which are local minima of the functional $\Phi - \lambda\Psi$ such that $\Psi(x_1) \geq 0$ and $\Psi(x_2) \geq 0$, one has

$$\inf_{t\in[0,1]} \Psi(tx_1 + (1-t)x_2) \geq 0;$$

(4) there exist a real Banach space $\widetilde{X}$ and a locally Lipschitz function $\widetilde{\Theta}_\lambda\colon \widetilde{X} \to \mathbb{R}$ such that X is compactly embedded in $\widetilde{X}$ and $\widetilde{\Theta}_\lambda\big|_X = \Theta_\lambda$.

We obtain the following result.

THEOREM 2.2. *Assume that there are $r_1, r_2 > 0$ and $\overline{v} \in K$, with $2r_1 < \Phi(\overline{v}) < r_2/2$, such that*

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Upsilon(u)}{r_1} < \frac{2}{3}\frac{\Upsilon(\overline{v})}{\Phi(\overline{v})}; \tag{2.7}$$

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Upsilon(u)}{r_2} < \frac{1}{3}\frac{\Upsilon(\overline{v})}{\Phi(\overline{v})}. \tag{2.8}$$

Furthermore, suppose that $\Phi(u) \geq r_2$ for all $u \in \partial K$. Then, for each $\lambda \in \overline{\Lambda}$, where $\overline{\Lambda}$ is given through

$$\overline{\Lambda} := \left]\frac{3}{2}\frac{\Phi(\overline{v})}{\Upsilon(\overline{v})}, \min\left\{\frac{r_1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Upsilon(u)}, \frac{r_2/2}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Upsilon(u)}\right\}\right[,$$

J_λ has three distinct critical points $u_i \in K$ $(i = 1, 2, 3)$ such that $\Phi(u_i) < r_2$ for $i \in \{1, 2, 3\}$.

Proof. Put $\rho_1 = r_1$ and $\rho_2 = r_2/2$. Because of $0 \in K$ and the definition of j, one has

$$\varphi_1(\rho_1) \leq \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,\rho_1[)} \Psi(u)}{\rho_1} \leq \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Upsilon(u)}{r_1}$$

and

$$\varphi_1(\rho_2) \leq \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,\rho_2[)} \Psi(u)}{\rho_2} \leq \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Upsilon(u)}{r_2/2}.$$

On the other hand, since (2.7) implies $\sup\limits_{u\in\Phi^{-1}(]-\infty,\rho_1[)} \Psi(u) < \Psi(\overline{v})$ and thanks to $\Phi(u) \geq 0$ for all $u \in X$, we obtain

$$\inf_{u\in\Phi^{-1}(]-\infty,\rho_1[)} \sup_{v\in\Phi^{-1}([\rho_1,\rho_2[)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)} \geq \frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}.$$

Hence,

$$\max\{\varphi_1(\rho_1), \varphi_1(\rho_2)\} < \inf_{u\in\Phi^{-1}(]-\infty,\rho_1[)} \sup_{v\in\Phi^{-1}([\rho_1,\rho_2[)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)}.$$

Therefore, owing to [3, Theorem 3.1] the functional J_λ admits two local minima $u_1, u_2 \in K$ such that $\Phi(u_1) < \rho_1$ and $\rho_1 \leq \Phi(u_2) < \rho_2$.

Now put $\Upsilon_{r_2/2\lambda}(u) = \min\{\Upsilon(u), r_2/2\lambda\}$ and $\mathfrak{F}_\lambda(u) = \Phi(u) - \lambda\Upsilon_{r_2/2\lambda}(u) + \lambda j(u)$ for all $u \in X$. Since $\Phi - \lambda\Upsilon_{r_2/2\lambda}$ is coercive, then [35, Proposition 2.3] ensures that $\mathfrak{F}_\lambda$ satisfies the $(\mathrm{PS})_c$-condition for all $c \in \mathbb{R}$. Put $\overline{c} = \inf\limits_{\gamma\in\Gamma} \max\limits_{t\in[0,1]} \mathfrak{F}_\lambda(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1]) : \gamma(0) = u_1, \gamma(1) = u_2\}$. It follows that

$$\overline{c} \leq \max_{t\in[0,1]} \mathfrak{F}_\lambda(tu_1 + (1-t)u_2) \leq \max_{t\in[0,1]} (t\Phi(u_1) + (1-t)\Phi(u_2)) < \rho_2.$$

Since

$$\mathfrak{F}_\lambda(u) = \Phi(u) - \lambda\Upsilon_{r_2/2\lambda}(u) \geq \Phi(u) - \lambda r_2/2\lambda \geq r_2/2 > \overline{c}, \quad \text{for all } u \in \partial K,$$

we have that $\mathfrak{F}_\lambda^{\overline{c}} = \{x \in X : \mathfrak{F}_\lambda(x) \geq \overline{c}\}$ is closed (see [33, Lemma 2.1]). Therefore, [33, Theorem 4.2] ensures the existence of a critical point u_3 of $\mathfrak{F}_\lambda$ such that $\overline{c} = \mathfrak{F}_\lambda(u_3)$. We claim that $\Upsilon(u_3) < r_2/2\lambda$. Arguing by contradiction, we assume that $\Upsilon(u_3) \geq r_2/2\lambda$. So, one has $\mathfrak{F}_\lambda(u_3) = \overline{c} < r_2/2$, that is

$$\Phi(u_3) - \lambda\Upsilon_{r_2/2\lambda}(u_3) < r_2/2, \quad \Phi(u_3) - r_2/2 < r_2/2, \quad \Phi(u_3) < r_2.$$

Therefore, since $\lambda \in \overline{\Lambda}$ and, in particular,

$$\lambda < \frac{r_2/2}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Upsilon(u)},$$

we conclude that $\Upsilon(u_3) < r_2/2\lambda$. But this is a contradiction, so our claim is proved. It follows that u_3 is also a critical point of J_λ. □

In the following, we consider an equivalent norm on the space $W^{1,p}(\Omega)$ given by

$$\|u\|_{W^{1,p}(\Omega)} := \left(\int_\Omega (|\nabla u(x)|^p + q(x)|u(x)|^p)\, dx \right)^{1/p}, \tag{2.9}$$

(see for instance [38, Section 1.1.15]).

3. The case $p > N$

In this section we prove the existence of multiple solutions to problem (1.1) if $p > N$. We recall that if $p > N$, the space $W^{1,p}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$, that is, there exists a positive constant c such that

$$\|u\|_{C(\overline{\Omega})} \leq c\|u\|_{W^{1,p}(\Omega)}.$$

First, we suppose there exist two constants $a_1, a_2 > 0$ such that

$$\frac{\max\limits_{|t|\leq a_1}(-F(t))}{a_1^p} < \frac{1}{c^p\|q\|_{L^1(\Omega)}} \frac{(-F(a_2))}{a_2^p} \tag{3.1}$$

and take $\lambda \in \Lambda$, where Λ is given by

$$\Lambda := \left] \frac{\|q\|_{L^1(\Omega)}}{p\|\alpha\|_{L^1(\Omega)}} \frac{a_2^p}{(-F(a_2))}, \frac{1}{pc^p\|\alpha\|_{L^1(\Omega)}} \frac{a_1^p}{\max\limits_{|t|\leq a_1}(-F(t))} \right[. \tag{3.2}$$

Thanks to (3.1) we observe that the interval Λ is non-empty.

Put

$$\begin{aligned} \delta := \min \Bigg\{ & \frac{a_1^p - pc^p\lambda\|\alpha\|_{L^1(\Omega)} \max\limits_{|t|\leq a_1}(-F(t))}{pc^p\|\beta\|_{L^1(\Gamma)} \max\limits_{|t|\leq a_1}(-G(t))}, \\ & \frac{p\lambda\|\alpha\|_{L^1(\Omega)}(-F(a_2)) - a_2^p\|q\|_{L^1(\Omega)}}{p\|\beta\|_{L^1(\Gamma)} \min\{0, -G(a_2)\}} \Bigg\}, \end{aligned} \tag{3.3}$$

$$\widetilde{\delta} = \frac{1}{\max\left\{0, p\max\{\|\alpha\|_{L^1(\Omega)}, \|\beta\|_{L^1(\Gamma)}\}c^p \limsup\limits_{|\xi|\to+\infty} \dfrac{(-G(\xi))}{|\xi|^p}\right\}}$$

$$\overline{\delta} := \min\{\delta, \widetilde{\delta}\}.$$

Since $\lambda \in \Lambda$, a simple computation shows that $\delta > 0$.

If $\limsup\limits_{|\xi|\to+\infty} \frac{(-G(\xi))}{|\xi|^p} \leq 0$ and $\max\limits_{|t|\leq a_1}(-G(t)) = 0$ as well as $G(a_2) \geq 0$ we read $\overline{\delta} = +\infty$.

Our main result in this section is the following.

Theorem 3.1. *Let $\alpha \in L^1(\Omega)$, $\beta \in L^1(\Gamma)$ be two non-negative and non-zero functions. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function and put $F(\xi) = \int_0^\xi f(t)\,dt$ for every $\xi \in \mathbb{R}$. Assume that*

$$\text{(H1)}\quad \frac{\max\limits_{|t|\le a_1}(-F(t))}{a_1^p} < \frac{1}{c^p\|q\|_{L^1(\Omega)}}\frac{(-F(a_2))}{a_2^p};$$

$$\text{(H2)}\quad \limsup_{|\xi|\to+\infty}\frac{(-F(\xi))}{\xi^p} \le 0,$$

with positive constants a_1, a_2 satisfying $a_1 < a_2$ and $a_2 \in K$. Then, for each $\lambda \in \Lambda$, where Λ is defined in (3.2), and for each locally essentially bounded function $g\colon \mathbb{R} \to \mathbb{R}$, whose potential $G(\xi) = \int_0^\xi g(t)\,dt$ for every $\xi \in \mathbb{R}$ satisfies

$$\text{(H3)}\quad \limsup_{|\xi|\to+\infty}\frac{(-G(\xi))}{|\xi|^p} < +\infty,$$

there exists $\overline{\delta} > 0$ given by (3.3) such that, for each $\mu \in [0, \overline{\delta}[$, problem (1.1) has at least three distinct solutions.

Proof. Our aim is to apply Theorem 2.1. Let λ, g and μ be fixed satisfying the assumptions and let X be the space $W^{1,p}(\Omega)$ equipped with the norm $\|\cdot\|_{W^{1,p}(\Omega)}$.

We set, for any $u \in X$,

$$\begin{aligned}
&\Phi(u) := \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)}, \qquad j(u) := \begin{cases} 0 & \text{if } u \in K,\\ +\infty & \text{otherwise,}\end{cases}\\
&\Upsilon(u) := \int_\Omega \alpha(x)[-F(u(x))]\,dx + \frac{\mu}{\lambda}\int_\Gamma \beta(x)[-G(\gamma u(x)]\,d\sigma,\\
&\Psi(u) := \Upsilon(u) - j(u), \qquad J_\lambda(u) := \Phi(u) - \lambda\Psi(u).
\end{aligned}$$

Then

$$\begin{aligned}
J_\lambda(u) = \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} &- \lambda\int_\Omega \alpha(x)[-F(u(x))]\,dx\\
&- \mu\int_\Gamma \beta(x)[-G(\gamma u(x)]\,d\sigma + \lambda j(u).
\end{aligned}$$

Let $r = \frac{1}{p}(\frac{a_1}{c})^p$, then $\Phi(a_2) > r$, that means $r \in \left]\inf\limits_X \Phi, \sup\limits_X \Phi\right[= \left]0, \sup\limits_X \Phi\right[$. Let $v \in \Phi^{-1}(]-\infty, r[)$. Then due to $\frac{1}{p}\|v\|^p_{W^{1,p}(\Omega)} < r$ combined with $\|v\|_{C(\overline{\Omega})}$

$\leq c\|v\|_{W^{1,p}(\Omega)}$, we have $|v(x)| \leq a_1$ for every $x \in \Omega$. We obtain

$$\begin{aligned}\varphi_1(r) &\leq \frac{\sup\limits_{\|w\|^p \leq pr} \left(\int_\Omega \alpha(x)[-F(w(x))]\,dx + \frac{\mu}{\lambda} \int_\Gamma \beta(x)[-G(\gamma w(x))]\,d\sigma - j(w) \right)}{r} \\ &\leq \frac{\left(\|\alpha\|_{L^1(\Omega)} \max\limits_{|t|\leq a_1}(-F(t)) + \frac{\mu}{\lambda}\|\beta\|_{L^1(\Gamma)} \max\limits_{|t|\leq r}(-G(r)) \right)}{r} \\ &= pc^p\|\alpha\|_{L^1(\Omega)} \frac{\max\limits_{|t|\leq a_1}(-F(t))}{a_1^p} + pc^p \frac{\mu}{\lambda}\|\beta\|_{L^1(\Gamma)} \frac{\max\limits_{|t|\leq a_1}(-G(t))}{a_1^p}.\end{aligned}$$

On the other side we have

$$\frac{\Psi(a_2)}{\Phi(a_2)} \geq \frac{p\|\alpha\|_{L^1(\Omega)}}{\|q\|_{L^1(\Omega)}} \frac{[-F(a_2)]}{a_2^p} + \frac{\|\beta\|_{L^1(\Gamma)}\mu p}{\lambda\|q\|_{L^1(\Omega)}} \frac{\min\{0, -G(a_2)\}}{a_2^p}.$$

As $\mu < \delta$ (see (3.3)), it follows

$$\begin{aligned} &\mu < \frac{a_1^p - pc^p\lambda\|\alpha\|_{L^1(\Omega)} \max\limits_{|t|\leq a_1}(-F(t))}{pc^p\|\beta\|_{L^1(\Gamma)} \max\limits_{|t|\leq a_1}(-G(t))} \quad \text{and} \\ &\mu < \frac{p\lambda\|\alpha\|_{L^1(\Omega)}(-F(a_2)) - a_2^p\|q\|_{L^1(\Omega)}}{p\|\beta\|_{L^1(\Gamma)} \min\{0, -G(a_2)\}}. \end{aligned} \tag{3.4}$$

From (3.4) we obtain

$$\frac{pc^p\|\alpha\|_{L^1(\Omega)} \max\limits_{|t|\leq a_1}(-F(t))}{a_1^p} + \frac{\mu}{\lambda} \frac{pc^p\|\beta\|_{L^1(\Gamma)} \max\limits_{|t|\leq a_1}(-G(t))}{a_1^p} < \frac{1}{\lambda},$$

respectively,

$$\frac{p\|\alpha\|_{L^1(\Omega)}}{\|q\|_{L^1(\Omega)}} \frac{[-F(a_1)]}{a_2^p} + \frac{\|\beta\|_{L^1(\Gamma)}\mu p}{\lambda\|q\|_{L^1(\Omega)}} \frac{\min\{0, -G(a_2)\}}{a_2^p} > \frac{1}{\lambda}.$$

Hence

$$\varphi_1(r) < \frac{1}{\lambda} < \frac{\Psi(a_2)}{\Phi(a_2)}. \tag{3.5}$$

In particular, we obtain from the calculations above that

$$\frac{\sup\limits_{\Phi(x)<r} \Psi(x)}{r} \leq \frac{\sup\limits_{\Phi(x)\leq r} \Psi(x)}{r} < \frac{\Psi(a_2)}{\Phi(a_2)},$$

where $0 < r < \Phi(a_2)$. This leads to

$$\varphi_2(r) \geq \frac{\Psi(a_2) - \sup\limits_{\Phi(x)\leq r} \Psi(x)}{\Phi(a_2) - r} \geq \frac{\Psi(a_2) - r\dfrac{\Psi(a_2)}{\Phi(a_2)}}{\Phi(a_2) - r} = \frac{\Psi(a_2)}{\Phi(a_2)}. \tag{3.6}$$

Combining (3.5) and (3.6) guarantees

$$\varphi_1(r) < \frac{1}{\lambda} < \varphi_2(r).$$

Let us now prove that J_λ is coercive. Since $\mu < \overline{\delta}$ and due to (H3), there exists a constant $k > 0$ such that

$$\limsup_{|\eta|\to+\infty} \frac{\max_{|t|\le\eta}(-G(t))}{\eta^p} < k \quad \text{and} \quad \mu k < \frac{1}{\max\{\|\alpha\|_{L^1(\Omega)}, \|\beta\|_{L^1(\Gamma)}\} p c^p}.$$

Thus, we get an estimate of the form

$$-G(s) \le k s^p + b_1 \quad \text{for all } s \in \mathbb{R}, \tag{3.7}$$

with a non-negative constant b_1. Putting τ fixed such that

$$0 < \tau < \left(\frac{1}{\max\{\|\alpha\|_{L^1(\Omega)}, \|\beta\|_{L^1(\Gamma)}\} p c^p} - \mu k \right) \frac{1}{\lambda}$$

yields, with the aid of hypothesis (H2),

$$-F(s) \le \tau s^p + b_2, \quad \text{for all } s \in \mathbb{R}, \tag{3.8}$$

with b_2 being non-negative. From (3.7) and (3.8) applied on the functional J_λ it follows, for $u \in D(j)$ (otherwise we are done),

$$\begin{aligned} J_\lambda(u) &= \Phi(u) - \lambda\Psi(u) \\ &\ge \left[\frac{1}{p} - \lambda\tau c^p \max\{\|\alpha\|_{L^1(\Omega)}, \|\beta\|_{L^1(\Gamma)}\} - \mu k c^p \max\{\|\alpha\|_{L^1(\Omega)}, \|\beta\|_{L^1(\Gamma)}\} \right] \\ &\quad \times \|u\|^p_{W^{1,p}(\Omega)} - \lambda b_2 \|\alpha\|_{L^1(\Omega)} - \mu b_1 \|\beta\|_{L^1(\Gamma)}, \end{aligned}$$

where

$$\frac{1}{p} - \lambda\tau c^p \max\{\|\alpha\|_{L^1(\Omega)}, \|\beta\|_{L^1(\Gamma)}\} - \mu k c^p \max\{\|\alpha\|_{L^1(\Omega)}, \|\beta\|_{L^1(\Gamma)}\} > 0.$$

This proves the coercivity of J_λ. In order to prove the Palais–Smale condition for J_λ, we have to apply [35, Proposition 2.3]. Now we are able to apply Theorem 2.1 obtaining the existence of three distinct critical points of J_λ denoted by u_1, u_2 and u_3. Let $u := u_1$ be the first critical point of J_λ, then one has

$$(\Phi - \lambda\Upsilon)^\circ(u; v-u) + \lambda j(v) - \lambda j(u) \ge 0 \quad \text{for all } v \in X. \tag{3.9}$$

Clearly, from (3.9) we see at once that $u \in K$ (otherwise (3.9) fails). Hence it follows

$$(\Phi - \lambda\Upsilon)^\circ(u; v-u) \ge 0 \quad \text{for all } v \in K. \tag{3.10}$$

The left-hand side of (3.10) can be estimated using (2.1)

$$\Phi'(u; v-u) + \lambda(-\Upsilon)^\circ(u; v-u) \ge 0 \quad \text{for all } v \in K,$$

which means

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla(v(x)-u(x))\,dx + \int_\Omega q(x)|u(x)|^{p-2}u(x)(v(x)-u(x))\,dx + \lambda(-\Upsilon)^\circ(u,v-u)\geq 0, \tag{3.11}$$

for all $v\in K$. Applying again (2.1) combined with the formula (2) in [18, p. 77] leads to

$$\lambda(-\Upsilon)^\circ(u;v-u)\leq \lambda\int_\Omega \alpha(x)F^\circ(u(x);v(x)-u(x))\,dx + \mu\int_\Gamma \beta(x)G^\circ(\gamma u(x);\gamma v(x)-\gamma u(x))\,d\sigma. \tag{3.12}$$

Finally, from (3.11) and (3.12) we have

$$\begin{aligned}&\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla(v(x)-u(x))\,dx\\ &+\int_\Omega q(x)|u(x)|^{p-2}u(x)(v(x)-u(x))\,dx\\ &+\lambda\int_\Omega \alpha(x)F^\circ(u(x);v(x)-u(x))\,dx\\ &+\mu\int_{\partial\Omega}\beta(x)G^\circ(\gamma u(x);\gamma v(x)-\gamma u(x))\,d\sigma\geq 0\quad\text{for all } v\in K.\end{aligned}$$

This proves that $u=u_1$ is a solution of our problem (1.1). The same calculations can be done for u_2, respectively u_3, which completes the proof of the theorem.□

Let us give a simple example to Theorem 3.1.

Example 3.2. Let $N<p\leq 11$, let K be a closed convex subset of $W^{1,p}(\Omega)$ with $2\in K$ and let $q(x)\equiv\frac{1}{c^p|\Omega|}$ for all $x\in\Omega$. We define the function $f\colon\mathbb{R}\to\mathbb{R}$ by

$$f(t)=\begin{cases}2e^{-t^2}t & \text{if } t<-1 \text{ and } t>2,\\ -2e^{-t^2}t^{17}(9-t^2)-1 & \text{if } -1\leq t\leq 2.\end{cases}$$

Putting $a_1=1$ and $a_2=2$ we conclude

$$\frac{\max\limits_{|t|\leq a_1}[-F(t)]}{a_1^p}=\max_{|t|\leq 1}[e^{-t^2}t^{18}+t]=e^{-1}+1<\frac{e^{-2^2}2^{18}+2}{2^p}=\frac{[-F(a_2)]}{a_2^p}.$$

Hence, condition (H1) of Theorem 3.1 is satisfied and (H2) is obviously true. Let $g\colon\mathbb{R}\to\mathbb{R}$ be defined by

$$g(t)=\begin{cases}-t^q & \text{if } t\leq 0,\\ -2t^q & \text{if } t>0\end{cases}$$

with $q<p-1$. Then, assumption (H3) is also fulfilled. The application of Theorem 3.1 yields the existence of three nontrivial solutions to (1.1).

Let us now show a special version of Theorem 3.1 when K is a ball. First, we show that every solution of (1.1) is non-negative provided the functions f and g are non-positive along with the assumption that the closed convex set K has partially lattice structure.

Proposition 3.3. *Let $p > N$ and assume that $f(t) \leq 0$ and $g(t) \leq 0$ for almost all $t \in \mathbb{R}$. Let $u \in K$ be a solution of (1.1) such that $u^+ = \max\{u, 0\} \in K$. Then, $u(x) \geq 0$ for all $x \in \Omega$.*

Proof. Let $u \in K \subset W^{1,p}(\Omega)$ be a solution of (1.1). Since f and g are non-positive, it is clear that F and G are non-increasing. Hence, we have $F^\circ(\xi; \eta - \xi) \leq 0$ and $G^\circ(\xi; \eta - \xi) \leq 0$ for all $\xi, \eta \in \mathbb{R}$ satisfying $\eta - \xi \geq 0$. Therefore, for all $v \in K$ such that $v(x) - u(x) \geq 0$ for all $x \in \Omega$, it follows

$$\int_\Omega \lambda\alpha(x) F^\circ(u(x); v(x) - u(x))\, dx \leq 0,$$
$$\int_\Gamma \mu\beta(x) G^\circ(\gamma u(x); \gamma v(x) - \gamma u(x))\, d\sigma \leq 0.$$

Applying this to (1.1) yields

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla(v(x) - u(x))\, dx + \int_\Omega q(x)|u(x)|^{p-2}u(x)(v(x) - u(x))\, dx \geq 0,$$

for all $v \in K$ such that $v(x) - u(x) \geq 0$ for all $x \in \Omega$.

Now, let $A := \{x \in \Omega : u(x) < 0\}$. Since $u^+(x) \geq u(x)$ for all $x \in \Omega$ and thanks to $u^+ \in K$, we may choose $v = u^+ \in K$ as test function to derive

$$-\int_A |\nabla(u(x))|^p\, dx - \int_A q(x)|u(x)|^p\, dx \geq 0,$$

meaning $\|u\|_{W^{1,p}(A)} \leq 0$. This proves the non-negativity of u. □

Remark 3.4. Note that if K is the ball $B(0, M)$ with center 0 and radius $M > 0$ it clearly holds $u^+ \in K$ which can be easily seen from the estimate $|u^+(x)| \leq |u(x)|$ for all $x \in \Omega$.

Next, suppose the existence of three positive constants a_1, a_2, and a_3 such that

$$\frac{[-F(a_1)]}{a_1^p} < \frac{2}{3c^p\|q\|_{L^1(\Omega)}}\frac{[-F(a_2)]}{a_2^p}, \qquad \frac{[-F(a_3)]}{a_3^p} < \frac{1}{3c^p\|q\|_{L^1(\Omega)}}\frac{[-F(a_2)]}{a_2^p}.$$

Taking

$$\Lambda = \left]\frac{3\|q\|_{L^1(\Omega)}}{2p\|\alpha\|_{L^1(\Omega)}}\frac{a_2^p}{[-F(a_2)]}, \min\left\{\frac{1}{pc^p\|\alpha\|_{L^1(\Omega)}}\frac{a_1^p}{[-F(a_1)]}, \frac{1}{pc^p\|\alpha\|_{L^1(\Omega)}}\frac{a_3^p}{2[-F(a_3)]}\right\}\right[\tag{3.13}$$

we see that Λ is non-empty. Let $\lambda \in \Lambda$ and put the number

$$\delta = \min\left\{\frac{a_1^p - \lambda pc^p\|\alpha\|_{L^1(\Omega)}[-F(a_1)]}{pc^p\|\beta\|_{L^1(\Gamma)}[-G(a_1)]}, \frac{a_3^p - \lambda 2pc^p\|\alpha\|_{L^1(\Omega)}[-F(a_3)]}{2pc^p\|\beta\|_{L^1(\Gamma)}[-G(a_3)]}\right\}, \tag{3.14}$$

which is obviously positive. If K is the ball $B(0, M)$, we have the following result.

THEOREM 3.5. *Let $K = B(0, M)$ and let $\alpha \in L^1(\Omega), \beta \in L^1(\Gamma)$ be two non-negative and non-zero functions. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function such that $f(t) \le 0$ for almost all $t \in \mathbb{R}$. Put $F(\xi) = \int_0^\xi f(t)\,dt$ for every $\xi \in \mathbb{R}$ and suppose that*

$$\text{(H1)}\quad \frac{[-F(a_1)]}{a_1^p} < \frac{2}{3c^p\|q\|_{L^1(\Omega)}}\frac{[-F(a_2)]}{a_2^p};$$

$$\text{(H2)}\quad \frac{[-F(a_3)]}{a_3^p} < \frac{1}{3c^p\|q\|_{L^1(\Omega)}}\frac{[-F(a_2)]}{a_2^p},$$

with positive constants a_1, a_2 and a_3 satisfying

$$2^{1/p}a_1 < c\|q\|_{L^1(\Omega)}^{1/p}a_2 < \frac{1}{2^{1/p}}a_3, \tag{3.15}$$

with $a_2 \in K$ and $a_3 \le c\|q\|_{L^1(\Omega)}^{1/p}M$. Then, for each $\lambda \in \Lambda$, where Λ is defined in (3.13)*, and for each locally essentially bounded function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(t) \le 0$ for almost all $x \in \mathbb{R}$ and with $G(\xi) := \int_0^\xi g(t)\,dt$ for every $\xi \in \mathbb{R}$, there exists $\delta > 0$ given by* (3.14) *such that for each $\mu \in [0, \delta[$, problem* (1.1) *has at least three distinct solutions u_i $(i = 1, 2, 3)$ satisfying $0 \le u_i(x) < a_3$ for all $x \in \Omega$ and all $i \in \{1, 2, 3\}$.*

PROOF. The ideas of the proof are mainly based on the proof of Theorem 3.1 with the difference that we want to apply Theorem 2.2 instead of Theorem 2.1. Let Φ, j, Υ, Ψ and J_λ as in the proof of Theorem 3.1 and let $r_1 = \frac{1}{p}(\frac{a_1}{c})^p$. Then we conclude

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)}\Upsilon(u)}{r_1} \le pc^p\|\alpha\|_{L^1(\Omega)}\frac{[-F(a_1)]}{a_1^p} + pc^p\frac{\mu}{\lambda}\|\beta\|_{L^1(\Gamma)}\frac{[-G(a_1)]}{a_1^p}.$$

On the other side, for $r_2 = \frac{1}{p}(\frac{a_3}{c})^p$ we obtain

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)}\Upsilon(u)}{r_2} \le pc^p\|\alpha\|_{L^1(\Omega)}\frac{[-F(a_3)]}{a_3^p} + pc^p\frac{\mu}{\lambda}\|\beta\|_{L^1(\Gamma)}\frac{[-G(a_3)]}{a_3^p}.$$

By assumption we have $\mu < \delta$ which results in

$$\mu < \frac{a_1^p - \lambda pc^p\|\alpha\|_{L^1(\Omega)}[-F(a_1)]}{pc^p\|\beta\|_{L^1(\Gamma)}[-G(a_1)]} \quad\text{and}\quad \mu < \frac{a_3^p - \lambda 2pc^p\|\alpha\|_{L^1(\Omega)}[-F(a_3)]}{2pc^p\|\beta\|_{L^1(\Gamma)}[-G(a_3)]}.$$

Combining these estimates above yields

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)}\Upsilon(u)}{r_1}<\frac{1}{\lambda}\quad\text{and}\quad\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)}\Upsilon(u)}{r_2/2}<\frac{1}{\lambda}. \tag{3.16}$$

We observe that

$$\frac{\Upsilon(a_2)}{\Phi(a_2)}\geq\frac{p\|\alpha\|_{L^1(\Omega)}}{\|q\|_{L^1(\Omega)}}\frac{[-F(a_2)]}{a_2^p}.$$

Furthermore, since $\lambda\in\Lambda$, we have

$$\frac{3\|q\|_{L^1(\Omega)}}{2p\|\alpha\|_{L^1(\Omega)}}\frac{a_2^p}{[-F(a_2)]}<\lambda.$$

Hence, it follows

$$\frac{2}{3}\frac{\Upsilon(a_2)}{\Phi(a_2)}>\frac{1}{\lambda}. \tag{3.17}$$

Now, we see that from (3.16) and (3.17) the estimates in (2.7) and (2.8), respectively, are satisfied. Taking into account (3.15) a simple calculation leads to

$$2r_1<\Phi(a_2)<\frac{r_2}{2}.$$

Let $u\in\partial K$, then, due to $a_3\leq c\|q\|_{L^1(\Omega)}^{1/p}M$, we have

$$\Phi(u)=\frac{1}{p}M^p\|q\|_{L^1(\Omega)}\geq\frac{1}{p}\left(\frac{a_3}{c}\right)^p=r_2.$$

Now we may apply Theorem 2.2 obtaining the existence of three distinct critical points $u_i\in K$ $(i=1,2,3)$ of J_λ satisfying $\Phi(u_i)<r_2$ for all $x\in\Omega$ and $i=1,2,3$. Similar to the proof of Theorem 3.1 it can be easily shown that these critical points are solutions of our original problem (1.1). Thanks to Proposition 3.3 and Remark 3.4 we conclude that these solutions are non-negative and owing to the embedding $W^{1,p}(\Omega)\hookrightarrow C(\overline{\Omega})$, we see that $u_i(x)<a_3$ is satisfied for all $x\in\Omega$ and for $i=1,2,3$. This completes the proof of the theorem. □

We close this section with an application of Theorem 3.5.

Example 3.6. Let $p>N$ with $p\geq 2$ and denote by $K=B(0,M)$ the ball with center zero and radius $M\geq 4$ while $q(x)\equiv\frac{1}{c^p|\Omega|}$ for all $x\in\Omega$. Further, let $g\colon\mathbb{R}\to\mathbb{R}$ be a non-positive, locally essentially bounded function and let $f\colon\mathbb{R}\to\mathbb{R}$ be defined through

$$f(t)=\begin{cases}-h(t) & \text{if } t<0 \text{ and } t\geq 4,\\ -1 & \text{if } 0\leq t<1 \text{ and } 2\leq t<4,\\ -3\cdot 2^{2p}t^2 & \text{if } 1\leq t<2,\end{cases}$$

with a non-negative, locally essentially bounded function $h\colon \mathbb{R} \to \mathbb{R}$. Put $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, then we have

$$2^{1/p} a_1 < c\|q\|_{L^1(\Omega)}^{1/p} a_2 < \frac{1}{2^{1/p}} a_3, \quad a_3 \le c\|q\|_{L^1(\Omega)}^{1/p} M \quad \text{and} \quad a_2 \in K.$$

A simple calculation shows

$$\begin{aligned}
\frac{[-F(a_1)]}{a_1^p} &= \frac{-\displaystyle\int_0^{a_1} -1\,dt}{a_1^p} = a_1 = 1,\\
\frac{[-F(a_2)]}{a_2^p} &= \frac{\displaystyle\int_0^{a_1} 1\,dt + \int_{a_1}^{a_2} 3\cdot 2^{2p} t^2\,dt}{a_2^p} = \frac{1 + 2^{2p}(2^3 - 1)}{2^p} = 7\cdot 2^p + 2^{-p},\\
\frac{[-F(a_3)]}{a_3^p} &= \frac{\displaystyle\int_0^{a_1} 1\,dt + \int_{a_1}^{a_2} 3\cdot 2^{2p} t^2\,dt + \int_{a_2}^{a_3} 1\,dt}{a_3^p}\\
&= \frac{1 + 2^{2p}(2^3 - 1) + (4 - 2)}{4^p} = 2^{-2p} + 9.
\end{aligned}$$

Hence, due to $p \ge 2$, it results in

$$\begin{aligned}
\frac{[-F(a_1)]}{a_1^p} &= 1 < \frac{14}{3} 2^p + \frac{2}{3} 2^{-p} = \frac{2}{3}\frac{[-F(a_2)]}{a_2^p},\\
\frac{[-F(a_3)]}{a_3^p} &= 2^{-2p} + 9 < \frac{7}{3}\cdot 2^p + \frac{1}{3} 2^{-p} = \frac{1}{3}\frac{[-F(a_2)]}{a_2^p}.
\end{aligned}$$

Now we may apply Theorem 3.5 to obtain the existence of three distinct solutions u_i $(i = 1, 2, 3)$ to problem (1.1) which are bounded through $0 \le u_i(x) < 4$ for $i = 1, 2, 3$.

4. The case $p \le N$

In this section we study the case $p \le N$. From now on it is supposed that $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma)$ and there exist constants $b_1, b_2 \ge 0$ such that

$$|f(t)| \le b_1 + b_2 |t|^{s-1} \tag{4.1}$$

for all $t \in \mathbb{R}$ while $s \in \,]1, Np/(N-p)[$ if $p < N$ as well as $s \in \,]1, +\infty[$ if $p = N$. Thanks to the Sobolev embedding and the trace embedding there are positive constants c_s and C satisfying

$$\|u\|_{L^s(\Omega)} \le c_s \|u\|_{W^{1,p}(\Omega)}, \qquad \|u\|_{L^p(\Gamma)} \le C\|u\|_{W^{1,p}(\Omega)} \tag{4.2}$$

for all $u \in W^{1,p}(\Omega)$. Applying the constants b_1, b_2 from (4.1) along with the constants c_1 and c_s in (4.2) we put

$$K_1 := \|\alpha\|_{L^\infty(\Omega)} b_1 c_1 p^{1/p}, \qquad K_2 := \|\alpha\|_{L^\infty(\Omega)} b_2 c_s^s \frac{p^{s/p}}{s}. \tag{4.3}$$

Now we assume there are two constants $a_1, a_2 > 0$ fulfilling $a_1 < \left(\frac{\|q\|_{L^1(\Omega)}}{p}\right)^{1/p} a_2$ with $a_2 \in K$ such that

$$K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p} < \frac{p\|\alpha\|_{L^1(\Omega)}}{\|q\|_{L^1(\Omega)}} \frac{[-F(a_2)]}{a_2^p}. \tag{4.4}$$

Due to (4.4) we see at once that

$$\widetilde{\Lambda} := \left] \frac{\|q\|_{L^1(\Omega)}}{p\|\alpha\|_{L^1(\Omega)}} \frac{a_2^p}{[-F(a_2)]}, \frac{1}{K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p}} \right[$$

is non-empty. Furthermore, let $g\colon \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function and set $G(\xi) = \int_0^\xi g(t)\,dt$ for $\xi \in \mathbb{R}$. We suppose that

$$[-G(\xi)] \le b_3 |\xi|^p$$

for all $\xi \in \mathbb{R}$ with b_3 being a non-negative constant. Similar to the case $p > N$ we put

$$\begin{aligned}
\delta &:= \min\left\{ \frac{1 - \lambda\left(K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p}\right)}{\|\beta\|_{L^\infty(\Gamma)} b_3 C^p p}, \frac{1 - \lambda\left(\frac{p\|\alpha\|_{L^1(\Omega)}}{\|q\|_{L^1(\Omega)}} \frac{[-F(a_2)]}{a_2^p}\right)}{\frac{p\|\beta\|_{L^1(\Gamma)}}{\|q\|_{L^1(\Omega)}} \frac{\min\{0, -G(a_2)\}}{a_2^p}} \right\}, \\
\widetilde{\delta} &= \frac{1}{\max\{\|\alpha\|_{L^\infty(\Omega)}, \|\beta\|_{L^\infty(\Gamma)}\} p c_p^p b_3}, \\
\overline{\delta} &:= \min\{\delta, \widetilde{\delta}\},
\end{aligned} \tag{4.5}$$

with $\lambda \in \widetilde{\Lambda}$.

Now we can formulate the main result in this section.

Theorem 4.1. *Let $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma)$ be two non-negative, non-zero functions and let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function satisfying the subcritical growth in* (4.1). *Put $F(\xi) = \int_0^\xi f(t)\,dt$ for all $\xi \in \mathbb{R}$ and suppose that*

(H1) $K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p} < \frac{p\|\alpha\|_{L^1(\Omega)}}{\|q\|_{L^1(\Omega)}} \frac{[-F(a_2)]}{a_2^p}$ *with positive constants a_1, a_2 satisfying $a_1 < (\|q\|_{L^1(\Omega)}/p)^{1/p} a_2$ and $a_2 \in K$;*

(H2) $\limsup_{|\xi| \to +\infty} \frac{[-F(\xi)]}{|\xi|^p} \le 0.$

Then, for each $\lambda \in \widetilde{\lambda}$ and for each locally essentially bounded function $g\colon \mathbb{R} \to \mathbb{R}$ whose potential $G(\xi) = \int_0^\xi g(t)\,dt$, $\xi \in \mathbb{R}$, fulfills

(H3) $[-G(\xi)] \le b_3 |\xi|^p$ *for all $\xi \in \mathbb{R}$ with b_3 being non-negative,*

there exists $\overline{\delta}$ *stated in* (4.5) *such that, for each* $\mu \in [0, \overline{\delta}[$, *problem* (1.1) *possesses at least three distinct solutions.*

PROOF. Similar to the proof of Theorem 3.1 we are going to apply Theorem 2.1 getting the assertion. Let λ, μ and g be fixed satisfying the assumptions and as before, let $X := W^{1,p}(\Omega)$ be endowed with the norm $\|\cdot\|_{W^{1,p}(\Omega)}$ given in (2.9). Now, we set again, for any $u \in X$,

$$\begin{aligned}
&\Phi(u) := \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)}, \qquad j(u) := \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise,} \end{cases} \\
&\Upsilon(u) := \int_\Omega \alpha(x)[-F(u(x))]\,dx + \frac{\mu}{\lambda}\int_\Gamma \beta(x)[-G(\gamma u(x)]\,d\sigma, \\
&\Psi(u) := \Upsilon(u) - j(u), \qquad J_\lambda(u) := \Phi(u) - \lambda\Psi(u)
\end{aligned}$$

leading to

$$\begin{aligned}
J_\lambda(u) = \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} &- \lambda\int_\Omega \alpha(x)[-F(u(x))]\,dx \\
&- \mu\int_\Gamma \beta(x)[-G(\gamma u(x)]\,d\sigma + \lambda j(u).
\end{aligned}$$

First, we observe that $\varphi_1(r) \le (1/r)\sup_{\Phi(u)<r} \Upsilon(u)$. With the aid of assumption (H3) along with the subcritical growth on f and the embeddings in (4.2), we may estimate Υ through

$$\begin{aligned}
\Upsilon(u) &= \int_\Omega \alpha(x)[-F(u(x))]\,dx + \frac{\mu}{\lambda}\int_\Gamma \beta(x)[-G(\gamma u(x)]\,d\sigma \\
&\le \|\alpha\|_{L^\infty(\Omega)} b_1 \|u\|_{L^1(\Omega)} + \|\alpha\|_{L^\infty(\Omega)}\frac{b_2}{s}\|u\|^s_{L^s(\Omega)} + \frac{\mu}{\lambda}\|\beta\|_{L^\infty(\Gamma)} b_3 \|\gamma u\|^p_{L^p(\Gamma)} \\
&\le \|\alpha\|_{L^\infty(\Omega)} b_1 c_1 \|u\|_{W^{1,p}(\Omega)} + \|\alpha\|_{L^\infty(\Omega)}\frac{b_2}{s}c_s^s\|u\|^s_{W^{1,p}(\Omega)} \\
&\quad + \frac{\mu}{\lambda}\|\beta\|_{L^\infty(\Gamma)} b_3 C^p \|u\|^p_{W^{1,p}(\Omega)} \\
&\le \frac{K_1}{p^{1/p}}\|u\|_{W^{1,p}(\Omega)} + \frac{K_2}{p^{s/p}}\|u\|^s_{W^{1,p}(\Omega)} + \frac{\mu}{\lambda}\|\beta\|_{L^\infty(\Gamma)} b_3 C^p \|u\|^p_{W^{1,p}(\Omega)}.
\end{aligned}$$

Since $\Phi(u) < r$ is equivalent to $\|u\|_{W^{1,p}(\Omega)} < (pr)^{1/p}$, we obtain, by setting $r = a_1^p$,

$$\begin{aligned}
\varphi_1(r) &\le \frac{K_1 r^{1/p} + K_2 r^{s/p} + \dfrac{\mu}{\lambda}\|\beta\|_{L^\infty(\Gamma)} b_3 C^p rp}{r} \\
&= K_1\frac{1}{a_1^{p-1}} + K_2 a_1^{s-p} + \frac{\mu}{\lambda}\|\beta\|_{L^\infty(\Gamma)} b_3 C^p p.
\end{aligned}$$

On the other hand, we have

$$\frac{\Psi(a_2)}{\Phi(a_2)} \ge \frac{p\|\alpha\|_{L^1(\Omega)}}{\|q\|_{L^1(\Omega)}}\frac{[-F(a_2)]}{a_2^p} + \frac{p\|\beta\|_{L^1(\Gamma)}}{\|q\|_{L^1(\Omega)}}\frac{\mu}{\lambda}\frac{\min\{0, -G(a_2)\}}{a_2^p}.$$

Then, since $\mu < \delta$, there holds

$$\mu < \frac{1-\lambda\left(K_1\dfrac{1}{a_1^{p-1}}+K_2a_1^{s-p}\right)}{\|\beta\|_{L^\infty(\Gamma)}b_3C^pp} \quad \text{and} \quad \mu < \frac{1-\lambda\left(\dfrac{p\|\alpha\|_{L^1(\Omega)}}{\|q\|_{L^1(\Omega)}}\dfrac{[-F(a_2)]}{a_2^p}\right)}{\dfrac{p\|\beta\|_{L^1(\Gamma)}}{\|q\|_{L^1(\Omega)}}\dfrac{\min\{0,-G(a_2)\}}{a_2^p}}.$$

This yields

$$K_1\frac{1}{a_1^{p-1}}+K_2a_1^{s-p}+\frac{\mu}{\lambda}\|\beta\|_{L^\infty(\Gamma)}b_3C^pp<\frac{1}{\lambda},$$

and

$$\frac{p\|\alpha\|_{L^1(\Omega)}}{\|q\|_{L^1(\Omega)}}\frac{[-F(a_2)]}{a_2^p}+\frac{p\|\beta\|_{L^1(\Gamma)}}{\|q\|_{L^1(\Omega)}}\frac{\mu}{\lambda}\frac{\min\{0,-G(a_2)\}}{a_2^p}>\frac{1}{\lambda}.$$

Finally, from the estimates above, we conclude

$$\varphi_1(r)<\frac{1}{\lambda}<\frac{\Psi(a_2)}{\Phi(a_2)}.$$

As already mentioned in the proof of Theorem 3.1, it holds, in particular,

$$\frac{\sup\limits_{\Phi(x)<r}\Psi(x)}{r}\le\frac{\sup\limits_{\Phi(x)\le r}\Psi(x)}{r}<\frac{\Psi(a_2)}{\Phi(a_2)},\quad 0<r<\Phi(a_2).$$

Thus, we derive

$$\varphi_2(r)\ge\frac{\Psi(a_2)-\sup\limits_{\Phi(x)\le r}\Psi(x)}{\Phi(a_2)-r}\ge\frac{\Psi(a_2)-r\dfrac{\Psi(a_2)}{\Phi(a_2)}}{\Phi(a_2)-r}=\frac{\Psi(a_2)}{\Phi(a_2)}.$$

Combining these estimates yields

$$\varphi_1(r)<\frac{1}{\lambda}<\varphi_2(r).$$

Finally, we note that the functional J_λ is coercive and satisfies the Palais–Smale condition (cf. the proof of Theorem 3.1). Hence, the assumptions of Theorem 2.1 are satisfied which ensures the existence of three distinct critical points of J_λ. That these critical points are solutions of (1.1) can be shown using the same arguments as in the end of the proof of Theorem 3.1. That finishes the proof of the theorem. □

We conclude with an application of Theorem 4.1.

Example 4.2. Let m be a positive constant specified later and let $s \in]1, Np/(N-p)[$ if $p < N$ as well as $s \in]1, +\infty[$ if $p = N$. We define the

functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ through

$$f(t) = \begin{cases} -\widetilde{C} & \text{if } t < 0, \\ -1 - |t|^{s-1} & \text{if } 0 \le t \le m, \\ -\dfrac{(1+m^2)(1+m^{s-1})}{1+t^2} & \text{if } t > m, \end{cases}$$

$$g(t) = \begin{cases} -|t|^{p-1} & \text{if } t \le 1, \\ -2|t|^{p-1} & \text{if } t > 1, \end{cases} \tag{4.6}$$

with $\widetilde{C}$ being a positive constant. Hence, for every constant $m > 0$, the mapping f fulfills the subcritical growth in (4.1) with $b_1 = \max\{1, \widetilde{C}\}$ and $b_2 = 1$. With the aid of these constants we define the numbers K_1 and K_2 as in (4.3). Now, fix m such that

$$m > \max\left\{ \left(\frac{p}{\|q\|_{L^1(\Omega)}} \right)^{1/p}, \left(s \frac{K_1 + K_2}{p} \frac{\|q\|_{L^1(\Omega)}}{\|\alpha\|_{L^1(\Omega)}} \right)^{1/(s-p)} \right\}. \tag{4.7}$$

Thanks to condition (4.7) it is clear that inequality

$$1 < \left(\frac{\|q\|_{L^1(\Omega)}}{p} \right)^{1/p} m$$

holds true. Moreover, we have

$$\frac{-F(m)}{m^p} = \frac{m^{s-p}}{s} + \frac{1}{m^{p-1}} > \frac{K_1 + K_2}{p} \frac{\|q\|_{L^1(\Omega)}}{\|\alpha\|_{L^1(\Omega)}}.$$

Setting $a_1 = 1$ and $a_2 = m$ we see at once that condition (H1) in Theorem 4.1 is satisfied. Additionally, for $\xi_1 > m$ and $\xi_2 < 0$, we point out that

$$\frac{[-F(\xi_1)]}{|\xi_1|^p} = (1+m^2)(1+m^{s-1}) \frac{\arctan(\xi_1)}{|\xi_1|^p} \to 0 \quad \text{as } \xi_1 \to +\infty,$$

$$\frac{[-F(\xi_2)]}{|\xi_2|^p} = \widetilde{C} \frac{|\xi_2|}{|\xi_2|^p} \to 0 \quad \text{as } \xi_2 \to -\infty$$

ensuring that (H2) is fulfilled as well. Since assumption (H3) holds for $b_3 = 2/p$, Theorem 4.1 can be applied to problem (1.1) with the special data in (4.6) which yields the existence of three distinct solutions of (1.1).

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BOUNDARY VALUE PROBLEMS WITH NONSMOOTH POTENTIAL, CONSTRAINTS AND PARAMETERS

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ABSTRACT. The paper focuses on the existence of multiple solutions of variational- hemivariational inequalities depending on parameters and involving the p-Laplacian operator on a bounded domain $\Omega \subset \mathbb{R}^N$. The parameters relevant for the solvability are precisely estimated. In the previous works such results have been obtained by assuming that $N < p$. Here we treat the case $N > p$, which is the main novelty of our work. The paper contains results regarding positive solutions as well.

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1. INTRODUCTION

The theory of hemivariational and variational-hemivariational inequalities deals with various problems in the form of inequalities containing nonlinear discontinuities and constraints. A multitude of techniques have been developed to study this type of problems, among which we mention variational methods connected to nonsmooth critical point theory (see, e.g., [12], [13]). In this respect, nonsmooth versions of the variational principle of Ricceri [15] have been utilized to show the existence of multiple solutions for hemivariational and variational-hemivariational inequalities formulated as boundary value problems and depending on parameters. These results enable us to find estimates for the range of parameters where the corresponding problems have multiple solutions. We refer to [1], [2], [3], [7], [8], [11], [14] for recent results in this direction. All these results establish the existence of multiple solutions of variational-hemivariational inequalities depending on parameters and involving the p-Laplacian

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operator on a bounded domain $\Omega \subset \mathbb{R}^N$ under the key assumption $N < p$. The goal of the present paper is to obtain such a result when $N > p$.

We describe the problem that we study here. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $N \geq 2$ and a C^1-boundary $\partial\Omega$ and let $1 < p < N$. We denote by $W_0^{1,p}(\Omega)$ the usual Sobolev space consisting of the elements of $W^{1,p}(\Omega)$ with zero traces on $\partial\Omega$. Given a closed, convex set $K \subset W_0^{1,p}(\Omega)$, consider the following inequality problem with constraints in K and depending on a real parameter $\lambda > 0$: Find $u \in K$ such that

$$\begin{aligned} &\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot (\nabla v(x) - \nabla u(x))dx \\ &\quad + \lambda \int_\Omega \alpha(x) F^\circ(u(x); v(x) - u(x))dx \geq 0 \qquad \text{for all } v \in K. \end{aligned} \tag{1.1}$$

Problems like (1.1) are called variational-hemivariational inequalities. In (1.1) it is supposed that $\alpha \in L^1(\Omega)$ satisfies $\operatorname{ess\,inf}_{x\in\Omega} \alpha(x) > 0$ and F° stands for Clarke's generalized directional derivative of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ for which we assume the subcritical growth condition

$$|\xi| \leq b_1 + b_2|t|^{s-1} \quad \text{for all } t \in \mathbb{R},\ \xi \in \partial F(t), \tag{1.2}$$

with constants $b_1, b_2 \geq 0$ and $1 < s < p^* := \frac{Np}{N-p}$. The notation ∂F in (1.2) means the generalized gradient of F. We note that under assumption (1.2), the integrals in (1.1) are well defined.

Our main result, which is stated as Theorem 3.1, provides a precise interval for the parameter λ such that the corresponding problem (1.1) for such a λ admits at least three distinct weak solutions. The main novelty of this result is the fact that it holds in the case $N > p$. We point out that a natural choice for the set of constraints K in (1.1) is the cone of nonnegative elements of $W_0^{1,p}(\Omega)$, that is

$$K = \{u \in W_0^{1,p}(\Omega) : u \geq 0 \text{ for a.a. } x \in \Omega\}. \tag{1.3}$$

With the same data α and F as above, we also consider the (unconstrained) problem depending on a real parameter $\lambda > 0$: Find $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)dx \\ &\quad + \lambda \int_\Omega \alpha(x) F^\circ(u(x); v(x))dx \geq 0 \quad \text{for all } v \in W_0^{1,p}(\Omega). \end{aligned} \tag{1.4}$$

Problems as the one stated in (1.4) are called hemivariational inequalities. There is a fundamental difference between the problems (1.1) and (1.4), which consists in the fact that (1.1) is verified through the set of constraints K whose elements act as test functions, whereas in (1.4) we act with any element of $W_0^{1,p}(\Omega)$. We want to explore the connection between the two problems related to the nonnegative solutions. Namely, take in (1.1) as set of constraints K the set introduced in (1.3), so

a solution of (1.1) is a nonnegative function. The question that we address is under what conditions this nonnegative function is also a (nonnegative) solution of (1.4). The converse assertion is clearly true, but generally not the raised one because the test functions in (1.4) are in the whole space $W_0^{1,p}(\Omega)$. Our results on this topic are given in Theorems 4.1 and 4.2, where we show that, under a verifiable condition on the generalized gradient ∂F, the question above has a positive answer meaning that the nonnegative solutions of (1.1) (with K in (1.3)) become solutions of (1.4).

The rest of the paper is organized as follows. Section 2 introduces the necessary mathematical background to be used later in the paper. Section 3 presents our main result on the multiple solutions of problem (1.1). Section 4 examines the connection between the nonnegative solutions of problems (1.1) and (1.4).

2. PRELIMINARIES

Let us start by recalling some basic notions in non-smooth analysis that are required in the sequel. For a real Banach space $(X, \|\cdot\|)$, we denote by X^* its dual space and by $\langle\cdot,\cdot\rangle$ the duality pairing between X and X^*. A function $f : X \to \mathbb{R}$ is said to be locally Lipschitz if for every $x \in X$ there exist a neighborhood U_x of x and a constant $L_x \geq 0$ such that

$$|f(y) - f(z)| \leq L_x \|y - z\| \quad \text{for all } y, z \in U_x.$$

For a locally Lipschitz function $f : X \to \mathbb{R}$ on a Banach space X, the generalized directional derivative of f at the point $x \in X$ along the direction $y \in X$ is defined by

$$f^\circ(x; y) := \limsup_{z \to x, t \to 0^+} \frac{f(z + ty) - f(z)}{t}$$

(see [6, Chapter 2]). If $f_1,\ f_2 : X \to \mathbb{R}$ are locally Lipschitz functions, then we have

$$(f_1 + f_2)^\circ(x, y) \leq f_1^\circ(x, y) + f_2^\circ(x, y) \quad \text{for all } x, y \in X. \tag{2.1}$$

The generalized gradient of a locally Lipschitz function $f : X \to \mathbb{R}$ at $x \in X$ is the set

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y\rangle \leq f^\circ(x; y) \ \text{ for all } y \in X\}.$$

An element $x \in X$ is said to be a critical point of a locally Lipschitz function $f : X \to \mathbb{R}$ if there holds

$$f^\circ(x; y) \geq 0 \quad \text{for all } y \in X$$

or, equivalently, $0 \in \partial f(x)$ (see [5]). More generally, for a function $I : X \to]-\infty, +\infty]$ expressed as $I = f + j$ with $f : X \to \mathbb{R}$ locally Lipschitz and $j : X \to]-\infty, +\infty]$ convex, lower semicontinuous function, $\not\equiv +\infty$, an element $u \in X$ is called a critical point of I if

$$f^\circ(u; v - u) + j(v) - j(u) \geq 0 \quad \text{for all } v \in X$$

is satisfied (see [12, Chapter 3] and [13]).

Our approach in studying problem (1.1) relies on an abstract three critical points theorem that we now describe. On a reflexive Banach space X, there are given a sequentially weakly lower semicontinuous and coercive function $\Phi : X \to \mathbb{R}$, a sequentially weakly upper semicontinuous function $\Upsilon : X \to \mathbb{R}$ and a convex, lower semicontinuous function $j : X \to]-\infty, +\infty]$ whose effective domain $D(j) = \{x \in X : j(x) < +\infty\}$ fulfills

$$D(j) \cap \Phi^{-1}(]-\infty, r[) \neq \emptyset \qquad \text{for all } r > \inf_X \Phi.$$

Set

$$\Psi := \Upsilon - j$$

and, for a real parameter $\lambda > 0$,

$$J_\lambda := \Phi - \lambda\Psi = (\Phi - \lambda\Upsilon) + \lambda j,$$

$$\varphi_1(r) = \inf_{y \in \Phi^{-1}(]-\infty, r[)} \frac{\left(\sup_{x \in \Phi^{-1}(]-\infty, r[)} \Psi(x)\right) - \Psi(y)}{r - \Phi(y)} \qquad \text{for all } r > \inf_X \Phi,$$

$$\varphi_2(r) = \sup_{y \in \Phi^{-1}(]r, +\infty[)} \frac{\Psi(y) - \left(\sup_{x \in \Phi^{-1}(]-\infty, r])} \Psi(x)\right)}{\Phi(y) - r} \qquad \text{for all } r < \sup_X \Phi.$$

The following result was recently proved in [3, Theorem 2.1].

Theorem 2.1. *Assume that there is* $r \in]\inf_X \Phi, \sup_X \Phi[$ *such that* $\varphi_1(r) < \varphi_2(r)$ *and the functional* J_λ *is bounded from below and satisfies the (PS)-condition for each* $\lambda \in \Lambda := \left]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right[$. *Then, for each* $\lambda \in \Lambda$, J_λ *has at least three distinct critical points.*

Such results originate in Ricceri's work (see [15] and the references therein). Nonsmooth versions can be found in [1] and [11].

3. MAIN RESULT

The Sobolev embedding theorem ensures the existence of a constant $c_{p^*} > 0$ such that

$$\|u\|_{L^{p^*}(\Omega)} \leq c_{p^*} \|u\|_{W_0^{1,p}(\Omega)} \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{3.1}$$

The expression of the best such constant is known

$$c_{p^*} = \frac{1}{\sqrt{\pi}} \frac{1}{N^{\frac{1}{p}}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(1+N-\frac{N}{p}\right)}\right)^{\frac{1}{N}}$$

(see Talenti [16]). As $s < p^*$ in (1.2), we have that

$$\|u\|_{L^s(\Omega)} \leq c_s \|u\|_{W_0^{1,p}(\Omega)} \quad \text{for all } u \in W_0^{1,p}(\Omega), \tag{3.2}$$

with a positive constant c_s which can be evaluated through Hölder's inequality and (3.1) as follows

$$c_s \leq |\Omega|^{\frac{p^*-s}{p^*s}} \frac{1}{\sqrt{\pi}} \frac{1}{N^{\frac{1}{p}}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(1+N-\frac{N}{p}\right)}\right)^{\frac{1}{N}}.$$

Denote

$$D := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega), \tag{3.3}$$

so it is clear that there is $x_0 \in \Omega$ such that the open ball $B(x_0, D)$ is contained in Ω. Using the positive constants b_1, b_2 in (1.2), c_1, c_s in (3.2) and D in (3.3), we set

$$\begin{aligned} K_1 &:= \frac{\|\alpha\|_{L^\infty(\Omega)} b_1 c_1 p^{\frac{1}{p}} p^{p-1}(p^N - (p-1)^N)}{(p-1)^N D^p}, \\ K_2 &:= \frac{\|\alpha\|_{L^\infty(\Omega)} b_2 c_s^s p^{\frac{s}{p}} p^{p-1}(p^N - (p-1)^N)}{s(p-1)^N D^p}, \end{aligned} \tag{3.4}$$

$$\kappa := \frac{D p^{\frac{1}{p}}}{p\pi^{\frac{N}{2p}}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)}{D^N - \left(\frac{(p-1)D}{p}\right)^N}\right)^{\frac{1}{p}}. \tag{3.5}$$

In view of the above remarks, the numbers K_1, K_2, κ in (3.4), (3.5) can be effectively estimated.

Now we state our main result.

Theorem 3.1. *Assume that $\alpha \in L^\infty(\Omega)$ fulfills $\operatorname{ess\,inf}_{x\in\Omega} \alpha(x) > 0$, $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, with $F(0) = 0$, satisfying the subcritical growth condition (1.2) and K is a nonempty, closed, convex subset of $W_0^{1,p}(\Omega)$. In addition, we suppose:*

(H1) $K_1 \dfrac{1}{a_1^{p-1}} + K_2 a_1^{s-p} < \operatorname*{ess\,inf}_{x\in\Omega} \alpha(x) \dfrac{[-F(a_2)]}{a_2^p}$ *for positive constants a_1, a_2 with $a_2 > \kappa a_1$ (see (3.5)), and $K_i, i = 1, 2$, given in (3.4);*

(H2) $\limsup\limits_{|\xi|\to+\infty} \dfrac{-F(\xi)}{|\xi|^p} \leq 0$;

(H3) $-F(t) \geq 0$ *for all $t \in [0, a_2]$;*

(H4) $u_{a_2} \in K$, *where*

$$u_{a_2}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, D), \\ \frac{pa_2}{D}(D - |x - x_0|) & \text{if } x \in B(x_0, D) \setminus B\left(x_0, \frac{(p-1)D}{p}\right), \\ a_2 & \text{if } x \in B\left(x_0, \frac{(p-1)D}{p}\right) \end{cases}$$

for all $x \in \Omega$. Then, for every $\lambda \in \Lambda$ with

$$\Lambda := \left] \frac{p^{p-1}(p^N-(p-1)^N)}{(p-1)^N D^p} \frac{1}{\operatorname{ess\,inf}_{x\in\Omega}\alpha(x)} \frac{a_2^p}{[-F(a_2)]}, \right. \\ \left. \frac{p^{p-1}(p^N-(p-1)^N)}{(p-1)^N D^p} \frac{1}{K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p}} \right[, \tag{3.6}$$

problem (1.1) *possesses at least three distinct weak solutions.*

Proof. Fix $\lambda \in \Lambda$. We are going to apply Theorem 2.1 on the space $X = W_0^{1,p}(\Omega)$. To this end, for any $u \in W_0^{1,p}(\Omega)$ we define

$$\Phi(u) := \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)}, \qquad j(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$\Upsilon(u) = \int_\Omega \alpha(x)[-F(u(x))]dx.$$

As in Section 1 we set

$$\Psi(u) = \Upsilon(u) - j(u), \qquad J_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

which results in

$$J_\lambda(u) = \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} - \lambda \int_\Omega \alpha(x)[-F(u(x))]dx.$$

Recall the function φ_1 in Section 2. Since $F(0) = 0$, it follows that

$$\varphi_1(r) \le \frac{\sup_{\Phi(u)<r} \Upsilon(u)}{r} \text{ for all } r > 0. \tag{3.7}$$

From (1.2), (3.2) and (3.4), we derive the estimate

$$\begin{aligned} \Upsilon(u) &\le \|\alpha\|_{L^\infty(\Omega)} b_1 \|u\|_{L^1(\Omega)} + \|\alpha\|_{L^\infty(\Omega)} \frac{b_2}{s} \|u\|^s_{L^s(\Omega)} \\ &\le \|\alpha\|_{L^\infty(\Omega)} b_1 c_1 \|u\|_{W_0^{1,p}(\Omega)} + \|\alpha\|_{L^\infty(\Omega)} \frac{b_2}{s} c_s^s \|u\|^s_{W_0^{1,p}(\Omega)} \\ &= \frac{(p-1)^N D^p K_1}{p^{\frac{1}{p}} p^{p-1}(p^N-(p-1)^N)} \|u\|_{W_0^{1,p}(\Omega)} \\ &\quad + \frac{(p-1)^N D^p K_2}{p^{\frac{s}{p}} p^{p-1}(p^N-(p-1)^N)} \|u\|^s_{W_0^{1,p}(\Omega)} \quad \text{for all } u \in W_0^{1,p}(\Omega). \end{aligned} \tag{3.8}$$

Then (3.7) and (3.8) yield

$$\varphi_1(a_1^p) \le \frac{(p-1)^N D^p}{p^{p-1}(p^N-(p-1)^N)} \left(K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p} \right). \tag{3.9}$$

By the definition of the function u_{a_2} we obtain

$$\begin{aligned}\Phi(u_{a_2}) &= \frac{1}{p}\int_{B(x_0,D)\setminus B\left(x_0,\frac{(p-1)D}{p}\right)} \frac{(pa_2)^p}{D^p}dx \\ &= \frac{1}{p}\frac{(pa_2)^p}{D^p}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left(D^N-\left(\frac{(p-1)D}{p}\right)^N\right).\end{aligned} \tag{3.10}$$

From hypothesis (H1) we know that $a_2 > \kappa a_1$, so (3.5) implies

$$\Phi(u_{a_2}) > a_1^p. \tag{3.11}$$

On the other hand, thanks to (H3) and (H4) we may write

$$\Psi(u_{a_2}) \geq -F(a_2)\operatorname*{ess\,inf}_{x\in\Omega}\alpha(x)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\frac{(p-1)^N D^N}{p^N}. \tag{3.12}$$

Combining (3.10) and (3.12) leads to

$$\frac{\Psi(u_{a_2})}{\Phi(u_{a_2})} \geq \frac{D^p}{p^{p-1}}\frac{(p-1)^N}{p^N-(p-1)^N}\operatorname*{ess\,inf}_{x\in\Omega}\alpha(x)\frac{[-F(a_2)]}{a_2^p}. \tag{3.13}$$

Now, on the basis of (H1), we obtain from (3.9) and (3.13) that

$$\varphi_1\left(a_1^p\right) < \frac{(p-1)^N D^p}{p^{p-1}(p^N-(p-1)^N)}\operatorname*{ess\,inf}_{x\in\Omega}\alpha(x)\frac{[-F(a_2)]}{a_2^p} \leq \frac{\Psi(u_{a_2})}{\Phi(u_{a_2})}. \tag{3.14}$$

Using (3.8), hypothesis (H1) and (3.13) ensures that

$$\begin{aligned}\frac{\sup_{\Phi(u)\leq a_1^p}\Psi(u)}{a_1^p} &\leq \frac{(p-1)^N D^p}{p^{p-1}(p^N-(p-1)^N)}\left(K_1\frac{1}{a_1^{p-1}}+K_2 a_1^{s-p}\right) \\ &< \frac{(p-1)^N D^p}{p^{p-1}(p^N-(p-1)^N)}\operatorname*{ess\,inf}_{x\in\Omega}\alpha(x)\frac{[-F(a_2)]}{a_2^p} \\ &\leq \frac{\Psi(u_{a_2})}{\Phi(u_{a_2})}.\end{aligned} \tag{3.15}$$

Let us recall the function φ_2 introduced in Section 2. Then (3.11) and (3.15) enable us to get

$$\varphi_2(a_1^p) \geq \frac{\Psi(u_{a_2})-\sup_{\Phi(u)\leq a_1^p}\Psi(u)}{\Phi(u_{a_2})-a_1^p} > \frac{\Psi(u_{a_2})-a_1^p\frac{\Psi(u_{a_2})}{\Phi(u_{a_2})}}{\Phi(u_{a_2})-a_1^p} = \frac{\Psi(u_{a_2})}{\Phi(u_{a_2})}. \tag{3.16}$$

It turns out from (3.14), (3.16) and (3.11) that

$$\varphi_1(a_1^p) < \varphi_2(a_1^p) \quad \text{and} \quad a_1^p \in]0, \Phi(u_{a_2})[.$$

Hence the assumption of Theorem 2.1 holds true for $r = a_1^p$.

Let us prove the coercivity of J_λ for every $\lambda \in \Lambda$ with Λ given in (3.6). Fix $\lambda \in \Lambda$ and a constant $\tau > 0$ satisfying

$$\tau < \frac{1}{\lambda p c_p^p \|\alpha\|_{L^\infty(\Omega)}}.$$

By (1.2) and (H2), we have the estimate

$$-F(t) \leq \tau |t|^p + \tilde{c} \quad \text{for all } t \in \mathbb{R} \tag{3.17}$$

with a constant $\tilde{c} \geq 0$. Then (3.17) and (3.2) with $s = p$ imply

$$J_\lambda(u) = \Phi(u) - \lambda \Psi(u) \geq \left[\frac{1}{p} - \lambda \tau c_p^p \|\alpha\|_{L^\infty(\Omega)} \right] \|u\|^p_{W^{1,p}(\Omega)} - \lambda \tilde{c} \|\alpha\|_{L^\infty(\Omega)},$$

for all $u \in W^{1,p}_0(\Omega)$. This shows that J_λ is coercive, so the Palais-Smale condition for J_λ follows (see [11, Proposition 2.3]). Therefore all the assumptions of Theorem 2.1 are fulfilled, so Theorem 2.1 with $r = a_1^p$ applies.

Gathering some estimates obtained in (3.16), (3.15) and (3.9) we infer that

$$\begin{aligned} \frac{1}{\varphi_2(a_1^p)} &< \frac{\Phi(u_{a_2})}{\Psi(u_{a_2})} \leq \frac{p^{p-1}(p^N - (p-1)^N)}{(p-1)^N D^p} \frac{a_2^p}{(\text{ess inf}_{x \in \Omega} \alpha(x))[-F(a_2)]} \\ &< \frac{p^{p-1}(p^N - (p-1)^N)}{(p-1)^N D^p} \frac{1}{K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p}} \leq \frac{1}{\varphi_1(a_1^p)}. \end{aligned}$$

It follows that the interval Λ obtained in Theorem 2.1 with $r = a_1^p$ contains the interval Λ in (3.6). We have thus shown that for any λ belonging to the interval Λ in (3.6) there exist at least three distinct critical points of J_λ.

Let u be a critical point of J_λ, which reads as

$$(\Phi - \lambda \Upsilon)^\circ(u; v - u) + \lambda j(v) - \lambda j(u) \geq 0 \quad \text{for all } v \in W^{1,p}_0(\Omega).$$

This amounts to saying that $u \in K$ and

$$(\Phi - \lambda \Upsilon)^\circ(u; v - u) \geq 0 \quad \text{for all } v \in K. \tag{3.18}$$

Thanks to (2.1), it is seen from (3.18) that

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot (\nabla v(x) - \nabla u(x)) dx + \lambda (-\Upsilon)^\circ(u; v - u) \geq 0 \quad \text{for all } v \in K. \tag{3.19}$$

We note that assumption (1.2) guarantees the applicability of formula (2) in [6, p. 77] to infer that

$$(-\Upsilon)^\circ(u; v - u) \leq \int_\Omega \alpha(x) F^\circ(u(x); v(x) - u(x)) dx. \tag{3.20}$$

Then (3.19) and (3.20) lead to

$$\begin{aligned} &\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot (\nabla v(x) - \nabla u(x)) dx \\ &\quad + \lambda \int_\Omega \alpha(x) F^\circ(u(x); v(x) - u(x)) dx \geq 0 \quad \text{for all } v \in K. \end{aligned}$$

This means exactly that u is a solution of problem (1.1). The proof is thus complete.

□

It is interesting that the interval Λ determined in Theorem 3.1 does not depend on the set of constrains K provided assumptions (H1)-(H4) are fulfilled. A meaningful case of Theorem 3.1 is when the closed convex set K is the cone of nonnegative functions.

Corollary 3.2. *Let the functions α and F be as in Theorem* 3.1 *and satisfy hypotheses* (H1)–(H3). *If $\lambda \in \Lambda$ for Λ in* (3.6), *then problem* (1.1) *with K in* (1.3) *possesses at least three distinct nonnegative weak solutions.*

Proof. Note that condition (H4) holds true for the set K in (1.3). Since the assumptions of Theorem 3.1 are satisfied, we may apply Theorem 3.1 which yields the stated result. □

4. NONNEGATIVE SOLUTIONS

In the setting of Corollary 3.2 we can compare the obtained solutions with the nonnegative solutions of the hemivariational inequality (1.4). Obviously, every nonnegative solution of (1.4) is a solution of (1.1) with K in (1.3). The next theorem addresses the converse assertion which occurs by strengthening hypothesis (H3).

Theorem 4.1. *Assume that $\alpha \in L^\infty(\Omega)$ satisfies* $\operatorname{ess\,inf}_{x\in\Omega} \alpha(x) > 0$ *and $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, with $F(0) = 0$, for which the subcritical growth condition* (1.2) *holds true and in addition*

(H3') *there is a constant $a_2 > 0$ such that*

$$\xi \le 0 \quad \text{for all } \xi \in \partial F(t) \text{ with } t \in [0, a_2]. \tag{4.1}$$

Then every solution $u \ge 0$ of problem (1.1), *with K in* (1.3) *and any $\lambda > 0$, that is*

$$\begin{aligned} &\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla(v(x) - u(x))dx \\ &\quad + \lambda \int_\Omega \alpha(x) F^\circ(u(x); v(x) - u(x))dx \ge 0 \end{aligned} \tag{4.2}$$

for all $v \in W_0^{1,p}(\Omega)$, $v \ge 0$, and which satisfies $u(x) \in [0, a_2]$ for all $x \in \Omega$, is a (nonnegative) solution of problem (1.4).

Proof. Let u be a solution of problem (1.1) with K in (1.3) satisfying $u(x) \in [0, a_2]$ for all $x \in \Omega$. We have to show that u is a solution of (1.4) if assumption (H3') is satisfied. To this end we follow an idea in the proof of Ma [10, Theorem 1.7]. Let $\varepsilon > 0$ and $w \in W_0^{1,p}(\Omega)$. Setting $w_\varepsilon = -\min\{0, u + \varepsilon w\}$, we have that $v = u + \varepsilon w + w_\varepsilon \in K$ (see (1.3)). Then (4.2) and (4.1), in conjunction with the subadditivity of $F^0(u; \cdot)$ and a basic property of the generalized gradient ∂F (see [6, p. 10]), yield

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla w(x)dx + \lambda \int_\Omega \alpha(x) F^\circ(u(x); w(x))dx$$

$$
\begin{aligned}
&\geq -\varepsilon^{-1}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla w_\varepsilon(x)dx - \varepsilon^{-1}\lambda\int_\Omega \alpha(x)F^\circ(u(x);w_\varepsilon(x))dx\\
&\geq \int_{\{u+\varepsilon w<0\}} |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla w(x)dx\\
&\quad - \varepsilon^{-1}\lambda\int_{\{u+\varepsilon w<0\}} \alpha(x)\max_{\xi\in\partial F(u(x))}\left[\xi(-u(x)-\varepsilon w(x))\right]dx\\
&= \int_{\{u+\varepsilon w<0\}} |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla w(x)dx\\
&\quad + \varepsilon^{-1}\lambda\int_{\{u+\varepsilon w<0\}} \alpha(x)\left(\max\partial F(u(x))\right)(u(x)+\varepsilon w(x))dx\\
&\geq \int_{\{u+\varepsilon w<0\}} |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla w(x)dx.
\end{aligned}
$$

Since $\nabla u = 0$ on $\{u = 0\}$ (see, e.g., [9, Lemma 7.7]), letting $\varepsilon \to 0$ in the inequality above leads to the conclusion that u solves problem (1.4). □

Next we set forth a further connection between the problems (1.1) and (4.2) which is related to a priori estimates of the solutions.

Theorem 4.2. *Assume that* $\alpha \in L^\infty(\Omega)$ *satisfies* $\operatorname{ess\,inf}_{x\in\Omega}\alpha(x) > 0$, $F : \mathbb{R}\to\mathbb{R}$ *is a locally Lipschitz function, with* $F(0) = 0$, *for which the subcritical growth condition* (1.2) *holds true, and in addition*

(H5) *there is a constant* $a_3 > 0$ *such that*

$$\max\partial F(t) \geq 0 \quad \textit{for all } t \geq a_3.$$

Then for every solution $u \geq 0$ *of problem* (1.1) *with* K *in* (1.3), *the following alternative hold: either* $\inf u < a_3$ *or* u *solves problem* (1.4).

Proof. Let u be a solution of problem (1.1) with K in (1.3) and some $\lambda > 0$ (equivalently, u solves problem (4.2)) such that $u \geq a_3$. For any $\varepsilon > 0$ and $w \in W_0^{1,p}(\Omega)$, taking the same test function $v = u + \varepsilon w + w_\varepsilon$ as in the proof of Theorem 4.2, the calculation therein gives

$$
\begin{aligned}
&\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla w(x)dx + \lambda\int_\Omega \alpha(x)F^\circ(u(x);w(x))dx\\
&\geq \int_{\{u+\varepsilon w<0\}} |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla w(x)dx\\
&\quad + \varepsilon^{-1}\lambda\int_{\{u+\varepsilon w<0\}} \alpha(x)\left(\max\partial F(u(x))\right)(u(x)+\varepsilon w(x))dx\\
&\geq \int_{\{u+\varepsilon w<0\}} |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla w(x)dx\\
&\quad + \lambda\int_{\{u+\varepsilon w<0\}} \alpha(x)\left(\max\partial F(u(x))\right)w(x)dx.
\end{aligned}
$$

The last inequality is valid due to hypothesis (H5) because $u \geq a_3 > 0$. Letting $\varepsilon \to 0$ in the inequalities above we derive that u is a solution of problem (1.4), which completes the proof. □

Remark 4.3. It is shown in [4] that, if the growth condition (1.2) holds with $s = p$, then (1.4) is equivalent to a differential inclusion problem involving the generalized gradient ∂F.

Remark 4.4. Hypothesis (H3') is a condition stronger than (H3) provided $F(0) \leq 0$. Indeed, by Lebourg's mean value theorem (see [6, Theorem 2.3.7]), for every $t \in (0, a_2]$ there is $\zeta \in (0, t)$ such that $F(t) = F(0) + \zeta t \leq \zeta t$. Thus (H3') implies (H3).

We provide an example for which all our hypotheses are satisfied, so all our results apply.

Example 4.5. Using the notation in Theorem 3.1 let us fix constants $a_1 > 0$, $a_2 > \kappa a_1$, $1 < r_1 < r_2 < p^*$ and

$$c > \max\left\{ a_2^{p-1} \left(\operatorname*{ess\,inf}_{x\in\Omega} \alpha(x) \right)^{-1} \left(K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p} \right) + \max\{a_2^{r_1-1}, a_2^{r_2-1}\}, \max\{r_1 a_1^{r_1-1}, r_2 a_2^{r_2-1}\} \right\}. \tag{4.3}$$

Define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(\xi) = \begin{cases} 0 & \text{if } \xi \leq 0 \\ -c\xi + \max\{\xi^{r_1}, \xi^{r_2}\} & \text{if } \xi > 0. \end{cases}$$

Then F is locally Lipschitz with the generalized gradient

$$\partial F(\xi) = \begin{cases} 0 & \text{if } \xi < 0 \\ [-c, 0] & \text{if } \xi = 0 \\ -c + r_1 \xi^{r_1-1} & \text{if } 0 < \xi < 1 \\ [-c + r_1, -c + r_2] & \text{if } \xi = 1 \\ -c + r_2 \xi^{r_2-1} & \text{if } \xi > 1. \end{cases}$$

The fact that (H1) is satisfied follows directly from (4.3) and the expression of F. Furthermore, we derive that

$$\limsup_{\xi\to+\infty} \frac{-F(\xi)}{\xi^p} = \limsup_{\xi\to+\infty} \left[\frac{c}{\xi^{p-1}} - \max\{\xi^{r_1-p}, \xi^{r_2-p}\} \right] \leq 0,$$

so (H2) holds true. Hypothesis (H3') is a straightforward consequence of the expression of ∂F and (4.3). In view of Remark 4.4, we infer that (H3) is valid. We have

already noted that (H4) is true for K in (1.3). In order to check (H5), it is sufficient to take any a_3 with $r_2 a_3^{r_2-1} > c$, which in particular ensures through (4.3) that $a_3 > a_2$.

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